A family of Riemann surfaces with orientation reversing automorphisms

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S Riemann surface $\Rightarrow$ S oriented surface.

$f : S \to S$ conformal automorphism $\Rightarrow$ f preserves orientation.

Many things are known on orientation preserving automorphisms.

- $\operatorname{ord}(f) \leq 4g+2$ (Wiman, Harvey).

- If $\operatorname{ord}(f) = 4g+2$ then
  - $\operatorname{Aut}^{+}(S) = \langle f \rangle$ (Kulkarni). But
  - $\operatorname{Aut}^{\pm}(S)$ is strictly larger than $\langle f \rangle$. In fact, S is symmetric (Singerman).

A *symmetry* is an orientation reversing automorphism of order 2.

- If $\operatorname{ord}(f) = 4g$ then
  - $\langle f \rangle \subsetneq \operatorname{Aut}^{+}(S)$ (Kulkarni). Furthermore,
  - S is symmetric (Singerman).

Less things are known on orientation reversing automorphisms.

- $\text{ord}(f) \leq \begin{cases} 4g+4 & \text{if } g \text{ is even,} \\ 4g-4 & \text{if } g \text{ is odd.} \end{cases}$ (Etayo).

- If the bound is attained then (Bagiński-Gromadzki).
  - $\langle f \rangle \subsetneq \text{Aut}^{\pm}(S)$ and
  - S is symmetric.

From now on

1) S is a compact Riemann surface of even genus g, and

2) $f : S \to S$ is an orientation reversing automorphism of order $2g$.

Remark: f^g is not a symmetry.

Uniformization

$S = \mathbb{H}/\Gamma$ where

- $\mathbb{H}$ is the hyperbolic plane, and

- Γ is a *surface Fuchsian group*, that is a fixed point-free Fuchsian group.

If $G < \mathrm{Aut}^+(S)$ then $G = \Lambda/\Gamma$ for some $\Lambda < \mathrm{Isom}^+(\mathbb{H})$.
(Fuchsian groups).

If $G < \mathrm{Aut}^\pm(S)$ then $G = \Delta/\Gamma$ for some $\Delta < \mathrm{Isom}^\pm(\mathbb{H})$
(NEC groups).

Theorem. With the above notations, and for $g \neq 6, 12, 30$:

1) $\langle f \rangle = \Gamma^*/\Gamma$ where Γ^* has signature $(1; -; [2, 2, g]; \{-\})$.

2) S is hyperelliptic and f^g is its hyperelliptic involution.

Proof:

1) Riemann-Hurwitz $+$ Harvey's theorem for orientation reversing automorphisms (Etayo).

2) Calculate the signature of the Fuchsian group uniformizing $\langle f^g \rangle$ and use Maclachlan's characterization of hyperelliptic surfaces by means of Fuchsian groups.

Remark. If $\mathrm{Aut}^{\pm}(S) = \langle f \rangle$ then S is asymmetric.

Proposition. $g \neq 6, 12, 30$. If S is asymmetric and admits an orientation reversing automorphism f of order $2g$ then g is even and

$$\mathrm{Aut}^{\pm}(S) = \langle f \rangle.$$

<u>Proof:</u>

Check the list of full groups of automorphisms of asymmetric pseudo-real hyperelliptic Riemann surfaces, by Bujalance and Turbek.

Consequently,

$$S \text{ is asymmetric} \Leftrightarrow \langle f \rangle \text{ does not extend.}$$

Algebraic equations

Theorem. $g \neq 2, 6, 12, 30$. There exist r, $s \in (0, \infty)$ and $\theta \in [0, 2\pi/g)$ such that S is given by

$$w^2 = z(z^{g/2} - r^{g/2})(z^{g/2} + 1/r^{g/2})(z^{g/2} - \alpha^{g/2})(z^{g/2} + 1/\overline{\alpha}^{g/2})$$

where $\alpha = se^{i\theta}$. A formula for f is

$$f : (z, w) \mapsto \left(\frac{e^{2\pi i/g}}{\overline{z}},\ \frac{\overline{w}}{\overline{z}^{g+1}}\, e^{i(\theta g/2 + \pi/g)} \right).$$

<u>Proof:</u>

Wootton's method for finding defining equations for cyclic prime covers of the sphere which admit additional orientation preserving automorphisms, or

1) find an orientation reversing $\widehat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of order g;

2) calculate the $\langle \widehat{f} \rangle$-orbit of a point in $\alpha \in \widehat{\mathbb{C}}$;

3) calculate the equation of the hyperelliptic surface S which ramifies over the appropriate number of $\langle \widehat{f} \rangle$-orbits;

4) calculate the liftings of $\widehat{f}$ to automorphisms $f : S \to S$ and discard those with order $< 2g$.

Question: Are these surfaces symmetric?

If so, which values of r, s and θ yield symmetric surfaces?

Proposition. $g \neq 2, 4, 6, 8, 10, 12, 30$. S is asymmetric, and hence $\text{Aut}^{\pm}(S) = \langle f \rangle$, if and only if

$$\theta \neq 0, 2\pi/g, \quad r \neq s, \quad r \neq 1/s, \quad (s, \theta) \neq (1, \pi/g), \quad (r, \theta) \neq (1, \pi/g).$$

Proof:

Find an orientation reversing involution $\widehat{\eta} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widehat{\eta} \circ \widehat{f}$ and $\widehat{f}^2$ generate either a cyclic or a dihedral group, and see the effect of $\widehat{\eta}$ on the branch points of S.

Remark: For the exceptional values of r, s and θ we have

$$\mathrm{C}_{2g} \subsetneq \mathrm{Aut}^{\pm}(S).$$

However, such cyclic group C_{2g} is always unique.

Proposition. $g \neq 2, 4, 6, 8, 10, 12, 30$. The group $\langle f \rangle$ generated by an orientation reversing automorphism f of order $2g$ is unique within $\mathrm{Aut}^{\pm}(S)$.

Teichmüller spaces

In terms of NEC groups.

Seppälä and Sorvali: *Geometry of Riemann surfaces and Teichmüller spaces*, North-Holland, 1992, Section 4.7.

Macbeath and Singerman: *Spaces of subgroups and Teichmüller spaces*, Proc. London Math. Soc., 1975.

Let Λ be an NEC group. We define

$$\begin{aligned} R(\Lambda) \;=\; & \{\tau : \Lambda \to \mathrm{PGL}(2,\mathbb{R}) \text{ such that } \tau \text{ is a monomorphism and} \\ & \tau(\Lambda) \text{ is also an NEC group}\}. \end{aligned}$$

$\mathrm{Aut}(\mathrm{PGL}(2,\mathbb{R}))$ acts on $R(\Lambda)$ by left multiplication. The orbit space

$$T(\Lambda) = \frac{R(\Lambda)}{\mathrm{Aut}(\mathrm{PGL}(2,\mathbb{R}))}$$

is the *Teichmüller space* of Λ.

If $\Lambda = \Gamma$ is a surface Fuchsian group uniformizing S then $T(\Gamma)$ is the Teichmüller space of compact Riemann surfaces of genus g.

Fact 1: $T(\Lambda)$ is a complete metric space of finite dimension. Its dimension can be computed from the signature of Λ.

For instance, if $\Lambda = \Gamma^*$ with signature $(1; -; [2, 2, g]; \{-\})$ then

$$\dim T(\Gamma^*) = 3.$$

Fact 2: Each monomorphism $i : \Lambda_1 \to \Lambda_2$ induces (by composition) $T(i) : T(\Lambda_2) \to T(\Lambda_1)$ which is an isometric embedding.

There is also a well defined action of $\mathrm{Aut}(\Lambda)$ on $T(\Lambda)$, given by right multiplication, which is not effective since $\mathrm{Inn}(\Lambda)$ acts trivially on $T(\Lambda)$. The factor group

$$M(\Lambda) = \frac{\mathrm{Aut}(\Lambda)}{\mathrm{Inn}(\Lambda)}$$

is the *modular group* of Λ.

Fact 3: $M(\Lambda)$ acts totally discontinuously on $T(\Lambda)$.

The orbit space

$$\frac{T(\Lambda)}{M(\Lambda)}$$

is the *moduli space* of Λ.

Let Γ^* be a fixed NEC group with signature $(1; -; [2, 2, g]; \{-\})$.

Γ^* contains a unique surface Fuchsian group Γ of index $2g$ with cyclic factor group.

Let

$$\begin{aligned} \mathsf{T}_{(2g)^-} = \{[\tau] \in \mathsf{T}(\Gamma) : \mathbb{H}/\tau(\Gamma) & \text{ has even genus } g \text{ and admits} \\ & \text{an orientation reversing automorphism } f \text{ of order } 2g\}. \end{aligned}$$

Theorem. $\mathsf{T}_{(2g)^-}$ is a three dimensional submanifold of $\mathsf{T}(\Gamma)$.

Proof:

Let $i : \Gamma \to \Gamma^*$ be the canonical inclusion. Then

$$T_{(2g)^-} = \bigcup_{\alpha \in \mathrm{Aut}(\Gamma)} \Big(\alpha \circ T(i)\Big)(T(\Gamma^*)).$$

This is a consequence of

- the characterization of $\langle f \rangle$ by means of NEC groups,
- the uniqueness of Γ in Γ^*.

Furthermore,

- The union is disjoint.

  This is a consequence of the uniqueness of $\langle f \rangle$ within $\mathsf{Aut}^{\pm}(S)$.

- The copies in the union do not accumulate in $T(\Lambda)$.

  This is a consequence of the totally discontinuous action of $M(\Lambda)$.

The proof follows from Fact 1 and Fact 2.