Beauville surfaces and finite groups

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1 Introduction

Algebraic geometers such as Bauer, Catanese and Grunewald have recently initiated the study of Beauville surfaces.

These are 2-dimensional complex algebraic varieties which are rigid, in the sense of admitting no deformations.

They are defined over the field $\overline{\mathbf{Q}}$ of algebraic numbers, and provide a geometric action of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$.

By generalising Beauville's original example, they can be constructed from finite groups acting on suitable pairs of algebraic curves.

Here we give some new examples of families of groups which can be used for this purpose.

2 Beauville surfaces

A *Beauville surface* (of unmixed type) is a compact complex surface $\mathcal{S}$ such that

(a) $\mathcal{S} \cong (\mathcal{C}_1 \times \mathcal{C}_2)/G$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are complex algebraic curves (= compact Riemann surfaces) of genus at least 2 and G is a finite group acting on each $\mathcal{C}_i$ so that it acts freely on $\mathcal{C}_1 \times \mathcal{C}_2$;

(b) $\mathcal{C}_i/G \cong \mathbf{P}^1(\mathbf{C})$ and the covering $\mathcal{C}_i \to \mathcal{C}_i/G$ is ramified over at most three points.

Condition (b) is equivalent to each curve $\mathcal{C}_i$ admitting a regular *dessin* in the sense of Grothendieck's theory of *dessins d'enfants*.

$\mathcal{S}$ is 2-dimensional over $\mathbf{C}$, so topologically it has dimension 4. Its fundamental group $\pi_1\mathcal{S}$ is an extension of $\pi_1\mathcal{C}_1 \times \pi_1\mathcal{C}_2$ by G.

A group G arises in this way if and only if it has generating triples (x_i, y_i, z_i) for $i = 1, 2$, of orders (l_i, m_i, n_i), such that

(1) $x_i y_i z_i = 1$ for each i,

(2) $l_i^{-1} + m_i^{-1} + n_i^{-1} < 1$ for each i, and

(3) no non-identity power of x_1, y_1 or z_1 is conjugate in G to a power of x_2, y_2 or z_2.

We call such a pair of triples (x_i, y_i, z_i) a *Beauville structure* on G.

Property (1) is equivalent to condition (a), with x_i, y_i and z_i representing the ramification over the three points.

Property (2) is equivalent to each $\mathcal{C}_i$ having genus at least 2 (arising as a smooth quotient of the hyperbolic plane).

Property (3) (which is always satisfied if $l_1m_1n_1$ is coprime to $l_2m_2n_2$) is equivalent to G acting freely on the product.

Beauville's original example (1978) used the Fermat curve $x^5 + y^5 + z^5 = 0$ of genus 6 for each $\mathcal{C}_i$, with $G = C_5 \times C_5$ acting in two different ways.

3 Conjecture and results

Bauer, Catanese and Grunewald (2006) have made the following conjecture:

Every non-abelian finite simple group except A_5 admits a Beauville structure.

They verified that the following simple groups admit Beauville structures

- the alternating groups A_n for all sufficiently large n,
- the projective special linear groups $L_2(p) = PSL_2(p)$ for all primes $p > 5$, and
- the Suzuki groups $Sz(2^e)$ for primes $e \geq 3$,

as do the following quasisimple groups (perfect central extensions of simple groups)

- the special linear groups $SL_2(p)$ for all primes $p > 5$.

Yolanda Fuertes and Gabino González-Diez (MZ, to appear) subsequently showed that A_n admits a Beauville structure if and only if $n \geq 6$. Here we will extend these results to show that the following families of groups have this property, namely

- $L_2(q)$ and $SL_2(q)$ for all prime powers $q > 5$,
- the Suzuki groups $Sz(2^e)$ for all odd $e \geq 3$, and
- the Ree groups $R(3^e)$ for all odd $e \geq 3$.

[Shortly after the AMS meeting we learnt that Garion and Penegini have recently obtained the same result about $L_2(q)$, and the results about the Ree and Suzuki groups for sufficiently large odd e, using different methods. See ArXiv:0910.5402.]

4 Strongly real Beauville structures

The Beauville structures on the groups $L_2(q)$ and $SL_2(q)$ for $q > 5$ can be chosen so that the corresponding Beauville surfaces are real.

A Beauville structure on G is *strongly real* if there are automorphisms α_i of G for $i = 1, 2$, differing by an inner automorphism, with each α_i inverting two elements of the triple (x_i, y_i, z_i). This implies that the corresponding Beauville surface $\mathcal{S}$ is real, that is, there is a biholomorphic involution $\sigma : \mathcal{S} \to \overline{\mathcal{S}}$. We can show that the following groups admit strongly real Beauville structures:

- $L_2(q)$ if and only if $q > 5$,
- $SL_2(q)$ if and only if $q > 5$.

[This is an updated and corrected version of the result presented at the AMS meeting, reflecting work done immediately after the meeting.]

5 Typical construction for $L_2(q)$

Suppose that $G = L_2(q)$ where $13 \leq q = p^e \equiv 1 \bmod (4)$, with p prime. Let

$$x_1 = \pm\begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}, \quad y_1 = \pm\begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad z_1 = (x_1 y_1)^{-1} = \pm\begin{pmatrix} 1 & 0 \\ b-a & 1 \end{pmatrix},$$

where $a, b \in F_q$ are chosen so that x_1 and y_1 have order $(q+1)/2$, and $a \neq b$ so that z_1 has order p. Let

$$x_2 = \pm\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad y_2 = \pm\begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad \text{and} \quad z_2 = (x_2 y_2)^{-1} = \pm\begin{pmatrix} c^{-1}w & -cy \\ -c^{-1}z & cx \end{pmatrix},$$

where c generates F_q^* and $xw - yz = 1$.

One can choose the entries of y_2 so that x_2, y_2 and z_2 have order $(q-1)/2$, coprime to the orders of x_1, y_1 and z_1, so condition (3) is satisfied. Provided $yz \neq 0$, Dickson's classification of the subgroups of $L_2(q)$ shows that neither triple is contained in any maximal subgroup, so each triple generates $L_2(q)$. Conditions (1) and (2) are clearly satisfied, so G admits a Beauville structure.

Conjugation by

$$\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in PGL_2(q)$$

inverts x_1, y_1 and x_2, and except for a few small q one can choose $z = -y$ so that it inverts y_2, giving a strongly real Beauville structure.

The constructions are similar for other $q > 5$, and for other groups such as $SL_2(q)$.