Fundamental Group of a Riemann Surface from Riemann Existence Theorem

Author: Yaacov Kopeliovich
email: ykopeliovich1@gmail.com

Introduction - Motivation

We all know how the fundamental group of a Riemann surface looks like:

Theorem

Let X be a Riemann surface. Then $\pi_1(X)$ is given by $2g$ generators subject to one relation:

$$\prod_{i=1}^{g}[a_i, b_i] = 1 \tag{1}$$

1. This theorem is a classical well known result for X Riemann surface so why am I interested in that?
2. ... because usually when we think about algebraic curves $f(x, y) = 0$ we think in terms of cuts and analytic continuation

The Problem

1. How do you get from cuts to the theorem above?
2. Of course we show this through topological arguments but I am looking for a direct proof.
3. I like to capture the Non Abelian part of the fundamental group as well.

Problem Formalizing

1. Given an equation $f(x, y) = 0$ it's well known that every x has n roots counting multiplicity
2. There is a finite number of points in $\lambda_1 ... \lambda_r \in \mathbb{CP}^1$ where we have less points on the fiber
3. $\lambda_1, ... \lambda_r$ are the ramification points.

Ramification Map

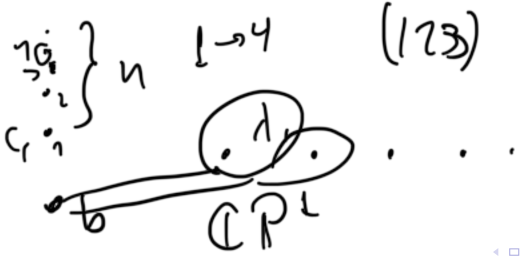
1. For each ramification point λ_i we have the preimages : $(\lambda_1, \psi_1), \ldots (\lambda_1, \psi_k)$
2. Each pre-image is given locally by the mapping $z \mapsto z^j$ and $\sum j = n$

Example

Let $y^3 = \prod_{i=1}^{r}(x - \lambda_i)$. In this case the ramification points are λ_i and the local coordinate is given by: $t = (x - \lambda_i)^{\frac{1}{3}}$ or $x = t^3 + \lambda_i$

Analytic Continuation

1. For a general equation $f(x, y) = 0$ the analytic continuation isn't as easy
2. The analytic continuation is encoded by picking a base point b on $\mathbb{CP}^1$ which has exactly n pre-images

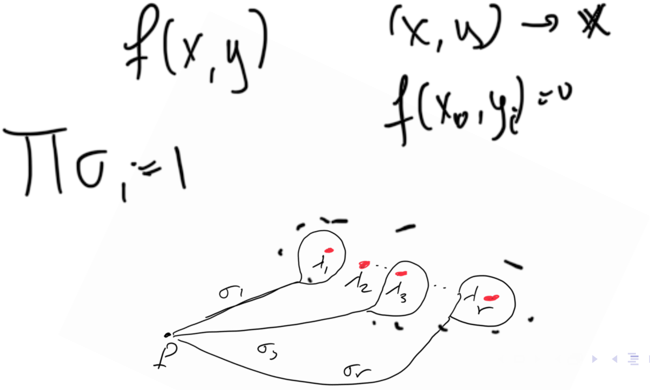


Analytic Continuation II

1. Create a loop surrounding the branch points and analytically continue across it
2. For the base point b select a pre-image b_j and lift σ_i to a pre-image starting with b_j
3. The loop σ_i will end on another pre-image of b, b_k and this induces a permutation representation for each σ_i.

The Loops on the $\mathbb{CP}^1 - \{\lambda_1, ...\lambda_r\}$

1. Schematic view of the loops is given below
2. σ_i rotates around the branch point and goes back to the base point on the $\mathbb{CP}^1$
3. $\prod_{i=1}^{r} \sigma_i = 1$ and $\pi_1(\mathbb{CP}^1 - \{\lambda_1, ...\lambda_r\}, b)$ can be regarded as a free group in generators $\sigma_1 ... \sigma_{r-1}$



Riemann Existence Theorem

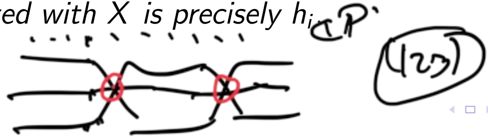
It turns out that the converse is also true and you can encode Algebraic curves via the permutation representation. We have the following theorem:

Theorem

Let $h_1, ...h_r$ be permutation in S_n satisfying the following 2 properties:

1. $\prod_{i=1}^{r} h_i = 1$
2. *h_i acts transitively on the set $\{1...n\}$*

Then there exists a compact algebraic curve X and a mapping $f : X \mapsto \mathbb{CP}^1$ such that the permutation representation associated with X is precisely h_i.

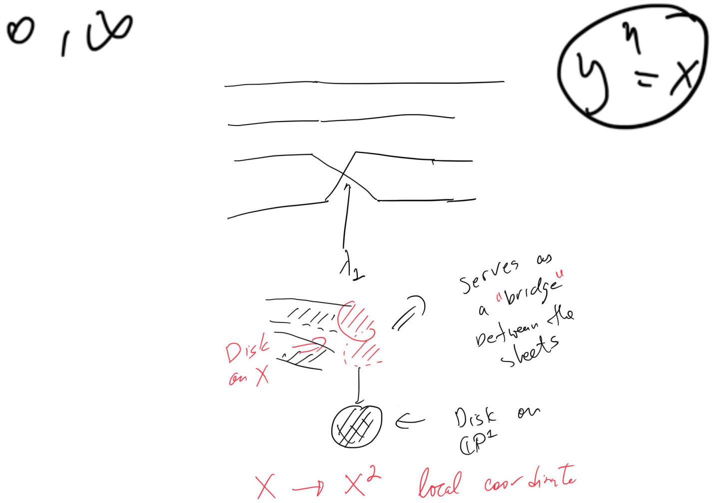


Proof of the theorem (sketch)

1. $\pi_1\left(\mathbf{CP}^1 \setminus (\lambda_1, ...\lambda_r)\right)$ is a group on r generators $\sigma_1, ...\sigma_r$ subject to the relation $\prod_{i=1}^{r} \sigma_i = 1$..
2. By covering theory $h_1, , ...h_r$ corresponds to a cover of $\pi_1\left(\mathbf{CP}^1 \setminus (\lambda_1, ...\lambda_r)\right)$. Call this cover X^{op}
3. To obtain X use the cycle decomposition of h_i to compactify X^{op}
4. More specifically if λ_i is a branch point then :
   4.1 Use the cycle decomposition of h_i to find the number of pre-images of h_i
   4.2 For each pre-image point the local mapping will be $x \mapsto x^{r_i}$ where r_i is the length of the cycle corresponding to this pre-image

$\sigma_i = (\quad)(\quad)(\quad)$

Proof of the theorem picture



The Fundamental group of $\pi_1(X)$

The classical Van Kampen theorem enables us to obtain $\pi_1(X)$:

Theorem

Assume that $h_i = \prod_{i=1}^{j} c_i$ and c_i are permutation cycles each one has length l_i. Then if $\pi_1(X^{op})$ is the fundamental group of the topological cover $\mathbf{CP}^1 \setminus (\lambda_1, \ldots\lambda_r)$ we have that

$$\pi_1(X) = {}^{\pi_1(X^{op})}/_{N} \tag{2}$$

N is the subgroup of $\pi_1(X^{op})$ generated by $\sigma_i^{l_i}$ if $\sigma_i^{l_i} \in \pi_1(X^{op})$ or: $\delta\sigma_i^{l_j}\delta^{-1} \in \pi_1(X^{op})$

$(123)(45)$

$\delta\sigma_1^{7}\delta^{-2}$ $\quad$ $\sigma_1^{3},$ $\quad$ $\mapsto h_1$

Generators for $\pi_1(X^{op})$

$\pi_1(X^{op}) = Stab(1)$

$X^{op} = f(x,y) - f^{-1}(\lambda_1 \dots \lambda_r) \to \mathbb{CP}^1 \setminus \{\lambda_1 \dots \lambda_r\}$

1. From the previous theorem to get $\pi_1(X)$ we need first to get generators $\pi_1\left(X^{op}\right)$.
2. Recall that $\pi_1(X^{op})$ is associated with the permutation representation of the open surface which is induced by $h_1, \dots h_r$.

Theorem

$\pi_1(X^{op})$ is the stabilizer of 1 (or any other number) in this permutation representation

$\sigma_i \in \pi_1(\mathbb{CP}^1 \setminus \{\lambda_1 \dots \lambda_r\}) \to \sigma_i \in S_n$

$\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1 \dots \lambda_r\}) \to S_n$

stab(1)

Schrier representatives of $\pi_1(X^{op})$

$\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1 \ldots \lambda_r\}) = \{\sigma_1 \ldots \sigma_r = 1\}$

$\sigma_1 \ldots \sigma_r \cdot 1$

Theorem

Regard $\{1, ..., n\}$ as co-sets of $\pi_1(\mathbb{CP}^1 - \{\lambda_1, ...\lambda_r\})/\pi_1(X)^{op}$. There are co-set representative R in with the following properties :

1. *The representative has the minimal length expressed through generators of $\pi_1(\mathbb{CP}^1 - \{\lambda_1, ...\lambda_r\})$ in all the words belonging to this co-set.*
2. *If $w = \prod_{1 \leq i_k \leq r} \sigma_{i_k}$ is a representative, than $\prod \sigma_{i_1} ... \sigma_{i_l}$ are representatives $i_l \leq r$*

(1234) (1) σ_1 σ_1^2 $\sigma_1 \ldots \sigma_r$ σ_1^{-1}

Scrhrier Generators

Theorem

Let $r_1, ...r_k$ *be the representatives from the theorem above For each* $r_i\sigma_l$ *we have a unique* r_m *such that* $h_{il} = r_i\sigma_l {r_m}^{-1} \in \pi_1(X^{op}, c_1)$ *than* h_{il} *are the generators for* $\pi_1(X^{op}, c_1)$

The case of (n, s) curves

1. These are curves that have at least one σ_r as a cyclic permutation of order n
2. An example is an equation $f(x, y)$ of the form:

$$f(x, y) = y^n - x^s - p(x, y) \tag{3}$$

and $deg_x p(x, y) < s$ and $deg_y p(x, y) < n$.

The relations $\sigma_i^k, i < r$

Theorem

Consider the relation σ_i^k or $\delta\sigma_i^j\delta^{-1}$. We can choose δ to be from R. Then we have the following theorem:

$$\sigma_i^j = \prod_{i=1}^{j} h_{ki} \tag{4}$$

such that the following is satisfied:

1. *Each generator appears exactly once in this product.*
2. *For two relations the intersection between the set of generators appearing in these relations is empty.*

Hence for each relation we can eliminate one generator using the relation $\prod h_{ki} = 1$.

The relation σ_r^n

What about the last relation? In the case we look on σ_r^n can be replaced by $\sigma_1 ... \sigma_{r-1}$ and hence we can write this relation as :

$$\sigma_1 \sigma_2 ... \sigma_{r-1} \sigma_1 ... \sigma_{r-1} ... = 1 \tag{5}$$

Lemma

The last relation can be written uniquely as :

$$\sigma_1 \sigma_2 ... \sigma_{r-1} \sigma_1 ... \sigma_{r-1} ... = \prod h_{ki} \tag{6}$$

where the product runs on all the generators of $\pi_1 \left(X^{op} \right)$ *.*

the Commutation word problem

Using the relations σ_i^l from previous slide we arrive to the following result:

Lemma

The relation

$$\sigma_1\sigma_2\ldots\sigma_{r-1}\sigma_1\ldots\sigma_{r-1}\ldots = \prod h_{ki} \tag{7}$$

is equivalent to a relation of the form:

$$\prod_{i=1}^{2g} s_1\ldots s_{2g}s_{i_1}^{-1}\ldots.s_{i_{2g}}^{-1} = 1$$

The commutation relation -part I

The last problem in the commutation word problem that was solved in an obscure paper by R.Goldstein and Turner (1979) using topological arguments. I modified their argumentation omitting the topology. I present the algorithm below:

1. First we can assume that our word is of the form:

$$x_1 R x_2 S x_1^{-1} T x_2^{-1} U.$$

   where R, S, T, U don't contain the generators x_1, x_2. We perform the following operations:

   1.1 Write :

$$x_1 R x_2 S x_1^{-1} T x_2^{-1} U = x_1 R x_2 T^{-1} TSRR^{-1} x_1^{-1} T x_2^{-1} U \quad (8)$$

   1.2 call $x_1 R = a$,and $x_2 T^{-1} = b$ to rewrite the last expression as:

$$abTSRa^{-1}b^{-1}U = abTSRa^{-1}b^{-1}(TSR)^{-1}TSRU \quad (9)$$

Commutation relation part II

In the previous slide define: $z = TSR$ and rewrite it as:

$$abza^{-1}b^{-1}z^{-1}zU \tag{10}$$

Then

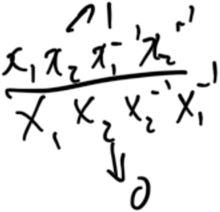
$$\begin{aligned} abza^{-1}b^{-1}z^{-1} \\ &= abb^{-1}z^{-1}za[za^{-1}, b^{-1}z^{-1}]zU \\ &= \left[za^{-1}, b^{-1}z^{-1}\right] zU \end{aligned} \tag{11}$$

Continue by induction eliminating the next pair of generators in zU using the same procedure.

The Intersection Form

While I have not shown it directly the last slide gives us a natural way to define the intersection form. For two generators we have the following combination of words:

1. $Yx_1Rx_2Sx_1^{-1}Tx_2^{-1}H$ - intersection 1
2. $Yx_1Rx_2Sx_2^{-1}Tx_1^{-1}H$ - intersection 0



Note: This is a speculation still requires proof

Example

Consider the curve given by the permutations (12345)...(12345) 5 times.

Lemma

The Schreier representatives are easy to find they are: $1, \sigma_1, \sigma_1^2, \sigma_1^{-1}, \sigma_1^{-2}$.

Using the recipe for the generators we have:

Lemma

The generators for $\pi_1(X^{op})$ *are:*

1. $s_{1i} = \sigma_i \sigma_1^{-1}$ *for* $i = 2, 3, 4$
2. $s_{2i} = \sigma_1 \sigma_i \sigma_1^{-2}$ *for* $i = 2, 3, 4$
3. $s_{3i} = \sigma_1^2 \sigma_i \sigma_1^2$ *for* $i = 1, 2, 3, 4$
4. $s_{4i} = \sigma_1^{-1} \sigma_i$ *for* $i = 2, 3, 4$
5. $s_{5i} = \sigma_1^{-2} \sigma_i \sigma_1$ *for* $i = 2, 3, 4$

The relations σ_i^5

We can easily write the relations $\sigma_i^5, i = 2, 3, 4$ in the following manner:

Lemma

For $i = 2, 3, 4$ we have that:

$$\sigma_i^5 = s_{1i} s_{2i} s_{3i} s_{5i} s_{4i} \tag{12}$$

and $\sigma_1^5 = 1$ kills the generator s_{31}

Theorem

Given the equation $y^5 = \prod_{i=1}^5 (x - \lambda_i)$ the non-normalized homology basis is: $s_{ij}, i = 1, 2, 3, 5,$ and $j = 2, 3, 4$

Thus the genus is 6 as required (I hope)

The relation is $(\sigma_1 ... \sigma_4)^5$

We can break this into the following words and write each one separately:

1. $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1 = s_{22}s_{33}s_{54}$
2. $\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2 = s_{12}s_{23}s_{34}s_{42}$
3. $\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3 = s_{13}s_{24}s_{31}s_{52}s_{43} = s_{13}s_{24}s_{52}s_{43}$
4. $\sigma_4\sigma_1\sigma_2\sigma_3\sigma_4 = s_{14}s_{32}s_{53}s_{44}$

and therefore we can write:

$$s_{22}s_{33}s_{53}s_{12}s_{23}s_{34}s_{42}s_{13}s_{24}s_{52}s_{43}s_{14}s_{32}s_{53}s_{44} = 1 \qquad (13)$$

Now we use the relation: $\sigma_i^5 = 1, i = 2, 3, 4$ to write:

1. $s_{44} = s_{54}^{-1}s_{34}^{-1}s_{24}^{-1}s_{14}^{-1}$
2. $s_{53} = s_{33}^{-1}s_{23}^{-1}s_{13}^{-1}s_{43}^{-1}$
3. $s_{32} = s_{22}^{-1}s_{12}^{-1}s_{42}^{-1}s_{52}^{-1}$

Thus we rewrite the last relation as:

$$s_{22}s_{33}s_{54}s_{12}s_{23}s_{34}s_{42}s_{13}s_{24}s_{52}s_{43}s_{14} \\ s_{22}^{-1}s_{12}^{-1}s_{42}^{-1}s_{52}^{-1}s_{33}^{-1}s_{23}^{-1}s_{13}^{-1}s_{43}^{-1}s_{54}^{-1}s_{34}^{-1}s_{24}^{-1}s_{14}^{-1} = 1 \quad (14)$$

The Intersection form

Note that we can read the intersection form. For example s_{22} with any other generator will be 1 however s_{54} with s_{12} will be 0. This is because we have the following word:
$Xs_{54}Ys_{12}Zs_{12}^{-1}Hs_{54}^{-1}U$

The normalized fundamental generators

We apply the technique developed to normalize the homology basis. In our case we have that:

1. $R_1 = 1$, $T_1 = s_{54}s_{12}s_{23}s_{34}s_{42}s_{13}s_{24}s_{52}s_{43}s_{14}$
2. $S_1 = s_{12}^{-1}s_{42}^{-1}s_{52}^{-1}$, $U_1 = s_{23}^{-1}s_{13}^{-1}s_{43}^{-1}s_{54}^{-1}s_{34}^{-1}s_{24}^{-1}s_{14}^{-1}$

if $z_1 = s_{54}s_{12}s_{23}s_{34}s_{42}s_{13}s_{24}s_{52}s_{43}s_{14}s_{12}^{-1}s_{42}^{-1}s_{52}^{-1}$ We have that the relation can be written as:

$$[zx_1^{-1}, x_2T_1^{-1}]s_{54}s_{12}s_{23}s_{34}s_{42}s_{13}s_{24}s_{52}s_{43}s_{14} \\ s_{12}^{-1}s_{42}^{-1}s_{52}^{-1}s_{23}^{-1}s_{13}^{-1}s_{43}^{-1}s_{54}^{-1}s_{34}^{-1}s_{24}^{-1}s_{14}^{-1} \quad (15)$$

... Continue in the same fashion

Conclusion and Discussion

1. The last slide gives an inductive algorithm to produce a_i, b_i such that $\prod_{i=1}^{g}[a_, b_i] = 1$
2. I haven't found this treatment in the literature.
3. This method enables us to define the intersection form in a purely formal way (no pictures like it's done usually)