Groups acting with almost all signatures

A. Wootton (based on work with S. Bozlee, M. Carvacho, J. Paulhus, T. Tucker)

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The Signature of a Group Action

Definition

Suppose that a finite group G acts on a compact oriented surface X of genus $\sigma \geq 2$. Then we define the signature of the action to be the tuple $(h; m_1, \ldots, m_r)$ where

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Signatures of Groups Examples

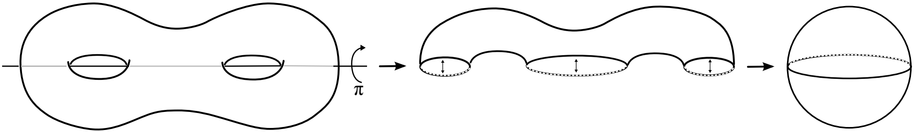

Figure: Cyclic 2 Action on Genus 2 Surface with Signature $(0; 2, 2, 2, 2, 2, 2)$

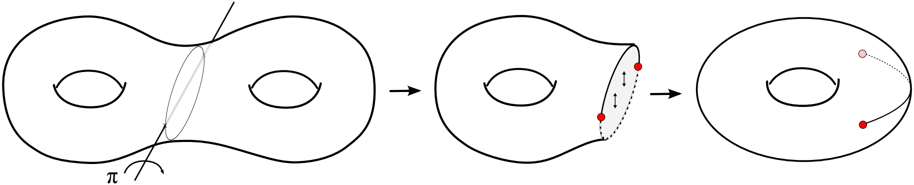

Figure: Cyclic 2 Action on Genus 2 Surface with Signature $(1; 2, 2)$

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Theorem

A finite group G acts on a compact Riemann surface X of genus $\sigma \geq 2$ with signature $(h; m_1, \ldots, m_r)$ if and only:

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Given a tuple $\mathcal{S} = (h; m_1, \ldots, m_r)$ *and a finite group* G*:*

- $\mathcal{S}$ *is called a* potential signature *for* G *if each* m_i *is the order of an element in* G*.*

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An Example: Finding Potential Signatures

$\mathcal{P}_9(C_6)$: tuples $(h; m_1, \ldots, m_r)$ with $r, h \geq 0$, where:

- $m_i \in \{2, 3, 6\}$

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$\mathcal{A}_9(S_3)$: tuples $(h; m_1, \ldots, m_r)$ from $\mathcal{P}_9(S_3)$ such that there exists a *generating vector* of group elements $(a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r)$ with:

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For a finite group G, finding potential signatures is easy, but finding actual signatures is difficult.

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