A series of graph curves with automorphisms

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Outline

1. Some problems I'm interested in
2. New developments
3. Graph curves
4. Example: a genus 7 graph curve
5. GIT
6. Higher genus analogues

Some problems I'm interested in

Describe birational models of $\overline{M}_g$ given by GIT quotients

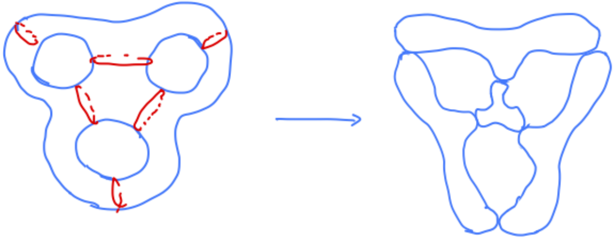
1. Models suggested by the Hassett-Keel program: quotients of
   - Hilbert schemes of bicanonical or canonical curves, small m
   - Grassmannian of syzygies of canonical curves
2. Mukai's models of $\overline{M}_7$, $\overline{M}_8$, $\overline{M}_9$

New developments

1. I have new software (`Sage`/`Macaulay2`) for studying graph curves
2. I identified a series of graphs with automorphisms with desirable properties
3. I noticed a new source of symmetry in state polytopes

The idea behind graph curves

Start with a smooth genus g curve. Degenerate as much as possible in $\overline{M}_g$.



Get a reducible curve.

- $2g - 2$ components
- $3g - 3$ nodes
- Each component has arithmetic genus 0
  and meets the rest of the curve in 3 nodes

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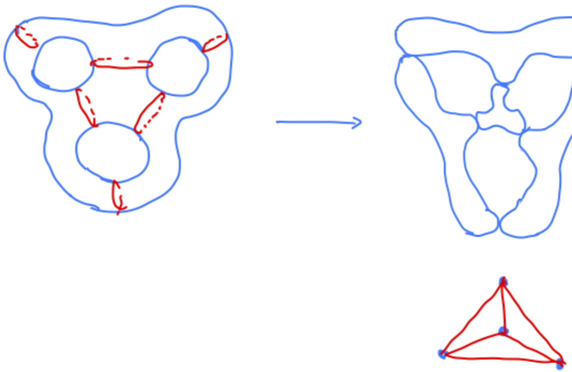

Draw the *dual graph* to record how these components intersect. (A vertex for each component, an edge for each node.)

Graph curves (in the sense of Bayer-Eisenbud)

Bayer-Eisenbud: A graph curve C is a connected, projective algebraic curve which is a union of projective lines, each meeting exactly three others, transversely at distinct points.

- Such a curve is determined up to isomorphism by its dual graph
- Graph curves are the 0-strata of the boundary of $\overline{M}_g$.

Selected results from [BE]

1. There are isomorphisms

$$H^0(C, \omega_C) \cong H^1(G, \mathbb{C})$$
$$H^0(C, \omega_C^2) \cong \mathrm{Coch}^1(G)$$
$$H^0(C, \omega_C^k) \cong \mathrm{Coch}^1(G) \oplus (H^1(G, \mathbb{C}))^{k-2} \quad \forall k \geq 2$$

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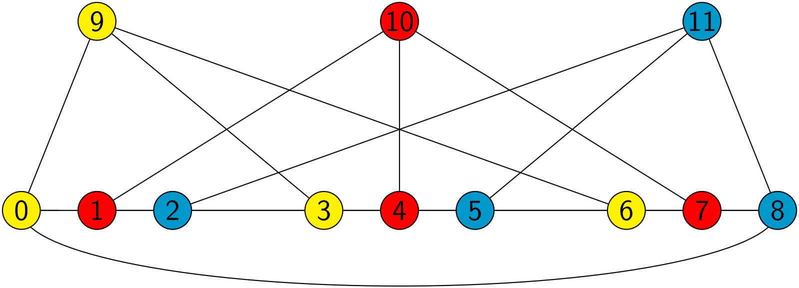
2. An algorithm for computing the canonical ideal

A genus 7 trivalent graph with 18 automorphisms



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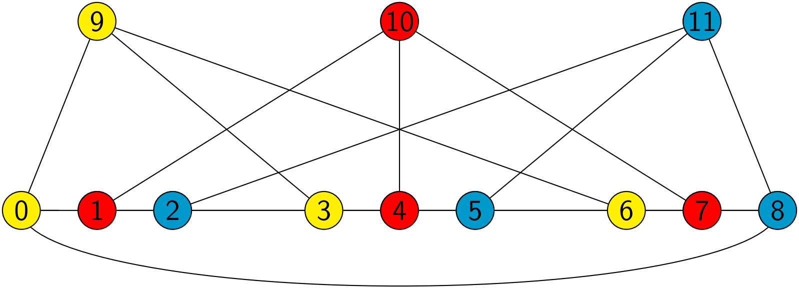

Automorphism group: the dihedral group of order 18.

Equations of C

Let z_i denote the 5-cycle starting at vertex i and moving counterclockwise. Let x_i be the cochain obtained by replacing edges in z_i by their dual basis vectors. Then we can identify $x_0, \ldots, x_6$ with a basis of $H^0(C, \omega)$.

Following [BE] we can get equations for the canonical ideal of C:

$$
\begin{aligned}
I = \langle & x_0x_4,\ x_0x_5,\ x_1x_5, \\
& x_1x_6,\ x_2x_6,\ x_2x_7, \\
& x_3x_7,\ x_3x_8,\ x_4x_8, \\
& x_3x_5 - x_4x_5 + x_4x_6 \rangle
\end{aligned}
$$

(Get a monomial x_ix_j if the cycles z_i and z_j are disjoint. Then rewrite x_7 and x_8 in the basis $x_0, \ldots, x_6$.)

Automorphism group action

The rotation σ acts on the basis $x_0, \ldots, x_6$ by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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The matrix for σ has distinct eigenvalues ζ_9^j for $j \in \{0, 1, 2, 4, 5, 7, 8\}$.
We have a *multiplicity-free* action on $H^0(C, \omega)$.

Geometric invariant theory (GIT)

GIT is a technique to construct the quotient of a scheme by the action of an algebraic group.

$X \subseteq \mathbb{P}^n$: projective scheme

G: reductive group acting on $\mathbb{P}^n$ by a representation $G \to \mathrm{GL}(n+1)$

$$X/\!/G := \mathrm{Proj}\left(k[x_0, \ldots, x_n]^G\right)$$

Issue: the quotient map $X \dashrightarrow X/\!/G$ has a base point at $x \in X$ if all G-invariant sections vanish at x.

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Definition $x \in X$ is *semistable* if there exists $s \in k[x_0, \dots, x_n]^G$ such that $s(x) \neq 0$.

Then there is a morphism

$$X^{ss} \to X/\!/G$$

Hilbert-Mumford numerical criterion

x is G-semistable $\iff$ x is λ-semistable for every 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$

$\iff$ x is T-semistable for every maximal torus $T \subset G$

$\iff$ the state polytope $\mathrm{St}_T(x)$ contains χ_0 $\forall$ maximal torus $T \subset G$

Kempf-Morrison-Swinarski Lemma

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Definition Let $X \subseteq \mathbb{P}^n$ be a projective scheme. Let $A \subseteq \mathrm{GL}(n+1)$ be a group of automorphisms. Decompose this into irreducible representations. The action of A on X is multiplicity-free if no irreducible representation has multiplicity greater than one.

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Lemma Suppose there is a group A whose action on X is multiplicity-free. Let $y_0, \ldots, y_n$ be coordinates on $\mathbb{P}^n$ adapted to the decomposition into irreducibles, and let T be the maximal torus scaling these coordinates. Then $[X]$ is G-semistable if and only if it is T-semistable.

Change of basis

By the KMS Lemma, the torus T scaling the eigenspaces of σ determines stability. Need to work in this basis. Call it $y_0, \dots, y_6$.

Write $\alpha_j = \zeta_9^j + \zeta_9^{-j}$

Weight	Quadrics
0	$3y_0^2 + (\alpha_1 - \alpha_2)y_2y_5 + (-\alpha_1 + \alpha_2 - 3)y_1y_6$,
	$y_3y_4 + (\alpha_4 - 1)y_2y_5 - \alpha_4 y_1y_6$
3	$y_1y_2 + (\alpha_4 - 1)y_3y_6 - \alpha_4 y_4y_5$
6	$y_2y_3 + (\alpha_1 - 1)y_5y_6 - \alpha_1 y_1y_4$
1	$3y_0y_1 - (\alpha_4 + 1)y_4^2 + (\alpha_4 - 2)y_2y_6$
2	$3y_0y_2 - (\alpha_1 + 1)y_1^2 + (\alpha_1 - 2)y_3y_5$
4	$3y_0y_3 - (\alpha_2 + 1)y_2^2 + (\alpha_2 - 2)y_4y_6$
5	$3y_0y_4 - (\alpha_2 + 1)y_5^2 + (\alpha_2 - 2)y_1y_3$
7	$3y_0y_5 - (\alpha_1 + 1)y_6^2 + (\alpha_1 - 2)y_2y_4$
8	$3y_0y_6 - (\alpha_4 + 1)y_3^2 + (\alpha_4 - 2)y_1y_5$

The Galois action

The eigenvalues of σ on $H^0(C, \omega_C)$ are 1 and the primitive ninth roots of unity.

The Galois group of the characteristic polynomial of σ permutes its roots, hence the eigenspaces, hence the variables $y_0, \ldots, y_6$.

$\mathrm{Gal}(\sigma)$ acts on I_2. We chose a basis of I_2 so that $\mathrm{Gal}(\sigma)$ permutes a subset of this basis, giving rise to a permutation on the corresponding state summands.

Example

The ζ_9 eigenspace of I_2 is generated by

$$3y_0y_1 - (\alpha_4 + 1)y_4^2 + (\alpha_4 - 2)y_2y_6$$

Then the $\mathrm{Gal}(\sigma)$ action $\zeta_9 \mapsto \zeta_9^2$ maps $y_0, \ldots, y_6$ to $y_0, y_2, y_3, y_6, y_1, y_4, y_5$, and maps this quadric to

$$3y_0y_2 - (\alpha_1 + 1)y_1^2 + (\alpha_1 - 2)y_3y_5.$$

This permutation of coordinates gives a bijection between the states for these two eigenspaces.

State polytope summands for I_2

The state polytope for I_2 is the Minkowski sum of the state polytopes in each weight.

Weight	States
0	(2, 1, 0, 0, 0, 0, 1), (2, 0, 1, 0, 0, 1, 0), (0, 1, 1, 0, 0, 1, 1), (2, 0, 0, 1, 1, 0, 0), (0, 1, 0, 1, 1, 0, 1), (0, 0, 1, 1, 1, 1, 0)
1	(1, 1, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0, 1), (0, 0, 0, 0, 2, 0, 0)
2	(1, 0, 1, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 1, 0)
3	(0, 1, 1, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 1), (0, 0, 0, 0, 1, 1, 0)
4	(1, 0, 0, 1, 0, 0, 0), (0, 0, 2, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0, 1)
5	(1, 0, 0, 0, 1, 0, 0), (0, 1, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 2, 0)
6	(0, 1, 0, 0, 1, 0, 0), (0, 0, 1, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1, 1)
7	(1, 0, 0, 0, 0, 1, 0), (0, 0, 1, 0, 1, 0, 0), (0, 0, 0, 0, 0, 0, 2)
8	(1, 0, 0, 0, 0, 0, 1), (0, 1, 0, 0, 0, 1, 0), (0, 0, 0, 2, 0, 0, 0)

GIT stability of the second Hilbert point

The trivial character is $\chi_0 = (\frac{20}{7}, \frac{20}{7}, \frac{20}{7}, \frac{20}{7}, \frac{20}{7}, \frac{20}{7}, \frac{20}{7})$

Want to show $\chi_0 \in \operatorname{St} I_2(C)$.

Strategy: construct two points ξ_0 and ξ_1 in the state polytope with $S_1 \times S_6$ symmetry such that

- ξ_0 overweights the first coordinate
- ξ_1 underweights the first coordinate

The point ξ_0

Weight	Summand of ξ_0
0	$\frac{1}{3}((2,1,0,0,0,0,1)+(2,0,1,0,0,1,0)+(2,0,0,1,1,0,0))$
1	$(1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0)$
2	$(1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)$
3	$\frac{1}{3}((0,1,1,0,0,0,0)+(0,0,0,1,0,0,1)+(0,0,0,0,1,1,0))$
4	$(1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0)$
5	$(1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0)$
6	$\frac{1}{3}((0,1,0,0,1,0,0)+(0,0,1,1,0,0,0)+(0,0,0,0,0,1,1)))$
7	$(1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0)$
8	$(1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1)$
ξ_0	$(8,2,2,2,2,2,2)$.

The point ξ_1

Weight	Summand of ξ_1
0	$\frac{1}{3}((0,1,1,0,0,1,1) + (0,1,0,1,1,0,1), +(0,0,1,1,1,1,0))$
1	(0, 0, 0, 0, 2, 0, 0)
2	(0, 2, 0, 0, 0, 0, 0)
3	$\frac{1}{3}((0,1,1,0,0,0,0) + (0,0,0,1,0,0,1) + (0,0,0,0,1,1,0))$
4	(0, 0, 2, 0, 0, 0, 0)
5	(0, 0, 0, 0, 0, 2, 0)
6	$\frac{1}{3}((0,1,0,0,1,0,0) + (0,0,1,1,0,0,0) + (0,0,0,0,0,1,1))$
7	(0, 0, 0, 0, 0, 0, 2)
8	(0, 0, 0, 2, 0, 0, 0)
ξ_1	$(0, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3})$

GIT stability of the second Hilbert point, and first syzygies

We have $\chi_0 = \frac{5}{14}\xi_0 + \frac{9}{14}\xi_1$

Then $\chi_0 \in \mathrm{St}\, I_2(C)$. Hence the second Hilbert point of C is T-semistable.

Since the torus T determines stability, the second Hilbert point of C is SL(7)-semistable.

Higher genus analogues for each odd $g \geq 5$

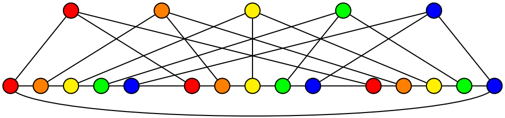
Write $g = 2k + 1$. Let G_k be the graph on $4k$ vertices with edges:

$e_{i,(i+1)}$ for $i = 0, \ldots, 3k - 1$;

$e_{0,3k-1}$;

$e_{i,3k+(i\%k)}$ for $i = 0, \ldots, 3k - 1$;

Let $C(G_k)$ be the associated graph curve.



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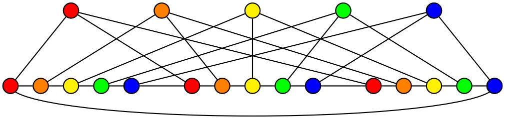

Lemma The action of the rotation of the $3k$-gon on $H^0(C(G_k), \omega)$ is multiplicity-free.

Theorem (S., 2024)

1. The second Hilbert point of $C(G_k)$ is $\mathrm{SL}(2k+1)$-semistable for $2 \leq k \leq 8$.
2. The first syzygy point of $C(G_k)$ is $\mathrm{SL}(2k+1)$-semistable for $k = 3, 4$.

Application: lighting