

Zero-sum multisets mod p with an application to surface automorphisms

Anthony Weaver
Bronx CC, CUNY

April 17, 2024

Zero-sum multisets $\leftrightarrow \mathbb{Z}_p^2$ actions

A zero-sum *multiset* M is an unordered collection (repeats allowed) of $R \geq 3$ non-zero columns from $V_2(F_p)$, such that

- ▶ M spans $V_2(F_p)$ (the *rank condition*);
- ▶ Both row sums $\equiv 0 \pmod{p}$ (the *zero-sum condition*).

Each such M serves as a generating vector for a *fully ramified* action of \mathbb{Z}_p^2 on a surface of genus

$$g = 1 + \frac{Rp(p-1)}{2} - p^2,$$

branched over R points on a surface of genus 0.

The projective marking

M can be ordered as a $2 \times R$ matrix in block form

$$M = [S_\infty \mid S_0 \mid S_1 \mid \dots \mid S_{p-1}].$$

S_i consists of $\mu_i \geq 0$ non-zero scalar multiples of the column \bar{i} .

$$\bar{i} \leftrightarrow \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad i = 0, \dots, p-1, \quad \overline{\infty} \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$\{\overline{\infty}, \overline{0}, \overline{1}, \dots, \overline{p-1}\} = PG(1, F_p)$. The tuple

$$\mathcal{P} = (\mu_\infty, \mu_0, \mu_1, \dots, \mu_{p-1})$$

is the *projective marking* induced by M .

The partition type

If the blocks of M are stacked vertically and reordered in weakly decreasing order of block size, the resulting array of columns yields the *Ferrers diagram* of \mathcal{P} , a.k.a., the *partition type* of M .

EXAMPLE: ($p = 5, R = 6$)

$$M = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{bmatrix} \quad \text{stacks as} \quad \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{array} \quad \begin{array}{c} \overline{\infty} \\ \overline{1} \\ \overline{4} \end{array}$$

has projective marking ($\mu_\infty = 3, \mu_1 = 2, \mu_4 = 1$) and partition type



The $GL_2(F_p)$ -action; orbit count

$GL_2(F_p)$ acts on zero-sum multisets, preserving

- ▶ rank
- ▶ zero sum
- ▶ *partition type*

$GL_2(F_p)$ orbits \leftrightarrow topological equivalence classes of \mathbb{Z}_p^2 actions.

DEF: $\mathcal{M}(\mathcal{P}) =$ set of zero-sum multisets of partition type \mathcal{P} .

- ▶ Counting $GL_2(F_p)$ orbits on R -multisets *reduces* to counting orbits on $\mathcal{M}(\mathcal{P})$, for each *admissible* partition \mathcal{P} of R .

$(R-1, 1)$ and (R) (e.g., for $R=6$, $\bullet\bullet\bullet\bullet\bullet$ and $\bullet\bullet\bullet\bullet\bullet\bullet$) are the *non-admissible* partition types.

The Lemma that is (not) Burnside's

For $\sigma \in GL_2(F_p)$ and $M \in \mathcal{M}(\mathcal{P})$, let

$$\text{Fix}_{\mathcal{P}}(\sigma) = \{M \in \mathcal{M}(\mathcal{P}) \mid \sigma M = M\}. \quad (1)$$

Then by Burnside's Lemma the number of orbits of $GL_2(F_p)$ on $\mathcal{M}(\mathcal{P})$ is

$$\frac{1}{|GL_2(F_p)|} \sum_{\sigma \in GL_2(F_p)} |\text{Fix}_{\mathcal{P}}(\sigma)|. \quad (2)$$

- ▶ When σ is the identity element, $|\text{Fix}_{\mathcal{P}}(\sigma)| = |\mathcal{M}(\mathcal{P})|$. This is the first non-trivial computation!

THEOREM 1 [W'24] For $\mathcal{P} = (1^{[s]})$, $3 \leq s \leq p + 1$,

$$|\mathcal{M}(\mathcal{P})| = \binom{p+1}{s} (p-1) \sum_{k=0}^{s-3} (-1)^k \binom{s-1}{k} p^{s-3-k}.$$

$$|\mathcal{M}(\mathcal{P})|, R = 6, p > 3$$

\mathcal{P}	$ \mathcal{M}(\mathcal{P}) $
$(1^{[6]})$	$(p+1)p(p-1)^2(p-2)(p-3)(p-4) \cdot (p^3 - 5p^2 + 10p - 10)/720$
$(2^{[1]}, 1^{[4]})$	$(p+1)p(p-1)^3(p-2)(p-3)(p^2 - 3p + 3)/48$
$(2^{[2]}, 1^{[2]})$	$(p+1)p(p-1)^5(p-2)/16$
$(3^{[1]}, 1^{[3]})$	$(p+1)^2p(p-1)^3(p-2)^2/36$
$(4^{[1]}, 1^{[2]})$	$(p+2)(p+1)^2p(p-1)^3/48$
$(3^{[1]}, 2^{[1]}, 1^{[1]})$	$(p+1)^2p(p-1)^4/12$
$(2^{[3]})$	$(p+1)p^2(p-1)^4/48$
$(3^{[2]})$	$(p+1)^3p(p-1)^2/72$
$(4^{[1]}, 2^{[1]})$	$(p+2)(p+1)^2p(p-1)^2/48$

Fixed points: $\sigma M = M$, $\sigma \in GL_2(F_p)$, $\sigma \neq \text{id}$

- ▶ If $\sigma M = M$ (up to column re-ordering), then M is a concatenation of σ -orbits on $V_2(F_p)$ (repeated orbits allowed).
- ▶ LEM: Any such concatenation automatically satisfies the zero-sum condition, **unless σ has an eigenvalue equal to 1**.

The elements of $GL_2(F_p)$

Geom. Type	Class rep.	No. Classes	No. Elts/Class
Central	$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, x \neq 0$	$p - 1$	1
Elliptic	$\begin{bmatrix} x & y\epsilon \\ y & x \end{bmatrix}, y \neq 0$	$\frac{p(p-1)}{2}$	$p^2 - p$
Parabolic	$\begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}, x \neq 0$	$p - 1$	$p^2 - 1$
Hyperbolic	$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, x \neq y$	$\frac{(p-1)(p-2)}{2}$	$p^2 + p$

Elements in $GL_2(F_p)$. $x, y \in F_p$, $\epsilon \in F_p^*$ a non-square

$$|GL_2(F_p)| = (p^2 - 1)(p^2 - p).$$

The central indicator

DEF: The *central indicator* of an element $\sigma \in GL_2(F_p)$ of order d is the smallest $d' \mid d$ such that $\sigma^{d'}$ is contained in the center of $GL_2(F_p)$. We put $d = d'd''$; then d'' is the *central quotient*.

- ▶ A *generic* σ -orbit on $V_2(F_p)$ spans d' subspaces, meeting each subspace d'' times.
- ▶ An *eigen-orbit* (if any) spans just one subspace (of course).

The shape of a σ -orbit

A generic σ -orbit can be displayed in the shape of a $d' \times d''$ array

$$X(\mathbf{v}, y) = \begin{bmatrix} \mathbf{v} & y\mathbf{v} & \dots & y^{d''-1}\mathbf{v} \\ \sigma(\mathbf{v}) & y\sigma(\mathbf{v}) & \dots & y^{d''-1}\sigma(\mathbf{v}) \\ \vdots & & & \\ \sigma^{d'-1}(\mathbf{v}) & y\sigma^{d'-1}(\mathbf{v}) & \dots & y^{d''-1}\sigma^{d'-1}(\mathbf{v}) \end{bmatrix},$$

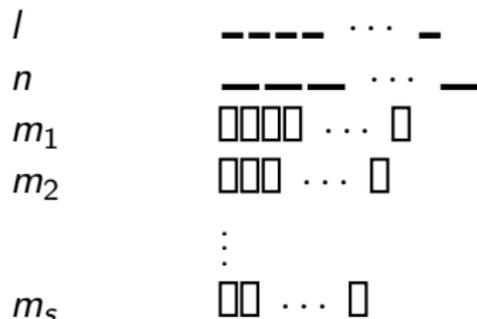
where

- ▶ $\mathbf{v} \in V_2(F_p)$
- ▶ $\sigma^{d'}$ is the central element $\begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$
- ▶ $y \in F_p^*$ has multiplicative order d''

(If \mathbf{v} were an eigenvector of σ , the array would have shape $1 \times d''$ and y would be an eigenvalue of multiplicative order d'' .)

Tiling the Ferrers diagram

If $\sigma M = M$, $M \in \mathcal{M}(\mathcal{P})$, the Ferrers diagram of \mathcal{P} can be overlaid by generic arrays (*tiles*) of shape $d' \times d''$, together (possibly) with some arrays of shape $1 \times e$ and/or $1 \times f$ confined to one or two rows. The tiled diagram looks like this:



Tile	Dimensions
—	$1 \times e$
—	$1 \times f$
□	$d' \times d''$

Class types in $GL_2(F_p)$

Within the four geometric types (central, elliptic, parabolic, hyperbolic) we define the more specific

DEF: class types

$${}_pC(d', d''), \quad {}_pE(d', d''), \quad {}_pP^e(d', d''), \quad {}_pH_f^e(d', d'')$$

which specify the central indicator d' , central quotient d'' , and *eigenorders* $e \leq f \in F_p^*$ (if any).

- ▶ Class types are smaller collections of conjugacy classes within the larger geometric types.
- ▶ Given p , which class types are non-empty, and how many classes do they contain?

Example: # of classes in each class type, $p = 7$

Central		Elliptic		Parabolic		Hyperbolic	
$C(1, d'')$	#	$E(d', d'')$	#	$P^e(p, d'')$	#	$H_f^e(d', d'')$	#
$C(1, 1)$	1	$E(2, 2)$	1	$P^1(7, 1)$	1	$H_6^6(3, 2)$	1
$C(1, 2)$	1	$E(2, 6)$	2	$P^2(7, 2)$	1	$H_6^3(6, 1)$	2
$C(1, 3)$	2	$E(4, 2)$	2	$P^3(7, 3)$	2	$H_6^3(2, 3)$	2
$C(1, 6)$	2	$E(4, 6)$	4	$P^6(7, 6)$	2	$H_6^2(3, 2)$	2
		$E(8, 2)$	4			$H_6^1(6, 1)$	2
		$E(8, 6)$	8			$H_3^2(6, 1)$	2
						$H_3^3(3, 1)$	1
						$H_3^1(3, 1)$	2
						$H_2^1(2, 1)$	1

The eigenvalue 1 cases

If σ has an eigenvalue 1, by change of basis, we may take σ to be

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in {}_p P^1(p, 1) \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} \in {}_p H_f^1(f, 1).$$

In the hyperbolic case, if $\sigma M = M$, $M \in \mathcal{M}(\mathcal{P})$, a tiled Ferrers diagram for \mathcal{P} has the form

l	● ● ● ● ⋯ ●	\bar{w}_1
n	— — — — ⋯ —	\bar{w}_f
m_1	⋯	$\langle \tilde{\sigma} \rangle \{ \bar{v}_1 \}$
m_2	⋯	$\langle \tilde{\sigma} \rangle \{ \bar{v}_2 \}$
	⋮	⋮
m_s	⋯	$\langle \tilde{\sigma} \rangle \{ \bar{v}_s \}$

$\text{Fix}_{\mathcal{P}}(\sigma), \sigma \in {}_p H_f^1(f, 1)$

THEOREM 4 [W'24] Let $\sigma \in {}_p H_f^1(f, 1)$. Let \mathcal{T} be a tiled Ferrers diagram overlying a partition \mathcal{P} which is feasible for σ . Then

$$|\text{Fix}_{\mathcal{T}}(\sigma)| = \frac{\left(\frac{p-1}{f}\right)!}{\left(\frac{p-1}{f} - s\right)! k_1! k_2! \dots k_u!} \binom{n + \frac{p-1}{f} - 1}{n} W_{(l, m_1, \dots, m_s)}.$$

where $W_{(l, m_1, \dots, m_s)}$ incorporates the (non-trivial) zero-sum condition.

Multiple tilings

Let \mathcal{P} be the partition underlying a tiling \mathcal{T} . We write $\mathcal{T} \supset \mathcal{P}$ in this situation. If there are two or more feasible tilings of a given partition and a given σ , then

$$|\text{Fix}_{\mathcal{P}}(\sigma)| = \sum_{\mathcal{T} \supset \mathcal{P}} |\text{Fix}_{\mathcal{T}}(\sigma)|.$$

For example, the partition $\mathcal{P} = (2^{[3]})$ has two feasible tilings for $\sigma \in {}_p H_2^1(2, 1)$ (for any odd p):

$$\mathcal{T}_1 = \ddot{\text{I}}, \quad \mathcal{T}_2 = \bar{\text{I}} \quad \supset \quad \mathcal{P} = \ddot{\text{I}}$$

Burnside's Lemma, adapted

Given \mathcal{P} , Burnside's Lemma is now reducible to a sum over representatives of the class types for which \mathcal{P} has a feasible tiling:

The number of orbits of $GL_2(F_p)$ on $\mathcal{M}(\mathcal{P})$ is

$$\frac{1}{(p^2 - 1)(p^2 - p)} \left\{ |\mathcal{M}(\mathcal{P})| + \sum_{\sigma \in F(\mathcal{P})} t(\sigma)c(\sigma) \sum_{\mathcal{T} \supset \mathcal{P}} |\text{Fix}_{\mathcal{T}}(\sigma)| \right\}$$

where

- ▶ $F(\mathcal{P}) \subset GL_2(F_p)$ contains one representative σ of each class type for which \mathcal{P} has a feasible tiling $\mathcal{T} \supset \mathcal{P}$;
- ▶ $c(\sigma)$ is the number of classes in the class type;
- ▶ $t(\sigma)$ is the number of elements in a class.

Example: $R = 6$, $\mathcal{P} = (2^{[3]})$, $p = 3, 5$

Burnside's Lemma computations:

\mathcal{T}	Class Type	Summand
$\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$	Id	12
$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$	${}_3C(1, 2)$	4
$\begin{array}{c} \bullet \\ \bullet \\ \text{---} \\ \text{---} \end{array}$	${}_3H_2^1(2, 1)$	36
$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$	${}_3H_2^1(2, 1)$	12
\square	${}_3P^2(3, 2)$	8
---	${}_3P^1(3, 1)$	24
Sum/48 =		96/48 = 2

$p = 3$

\mathcal{T}	Class Type	Summand
$\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$	Id	800
$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$	${}_5C(1, 2)$	160
$\begin{array}{c} \bullet \\ \bullet \\ \text{---} \\ \text{---} \end{array}$	${}_5H_2^1(2, 1)$	1200
$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$	${}_5H_2^1(2, 1)$	240
\square	${}_5E(3, 2)$	80
---	${}_5E(3, 1)$	400
Sum/480 =		2880/480 = 6

$p = 5$

\mathbb{Z}_3^2 actions on surfaces of genus 10 ($R = 6$)

Burnside calculations for all admissible 6-partitions yield a total of 10 topological types:

$$2^{[2]}1^{[2]} \quad 24; H_2^1(2, 1) \overset{\bullet\bullet}{\bar{1}} \quad 72; (24 + 72)/48 = \mathbf{2}.$$

$$3^{[1]}1^{[3]} \quad 16; P^1(3, 1) \overset{\bullet\bullet\bullet}{\bar{1}} \quad 32; (16 + 32)/48 = \mathbf{1}.$$

$$4^{[1]}1^{[2]} \quad 48; H_2^1(2, 1) \overset{\bullet\bullet\bullet}{\bar{1}} \quad 48; (48 + 48)/48 = \mathbf{2}.$$

$$3^{[1]}2^{[1]}1^{[1]} \quad 48; 48/48 = \mathbf{1}.$$

$$2^{[3]} \quad 12; C(1, 2) \equiv 4; P^1(3, 1) \parallel 24; P^2(3, 2) \blacksquare 8;$$
$$H_2^1(2, 1) \overset{\bullet\bullet}{\bar{1}\bar{1}} \quad 36; H_2^1(2, 1) \bar{1}\bar{1} \quad 12;$$
$$(12 + 4 + 24 + 8 + 36 + 12)/48 = \mathbf{2}.$$

$$3^{[2]} \quad 24; H_2^1(2, 1) \text{III} \quad 24; (24 + 24)/48 = \mathbf{1}.$$

$$4^{[1]}2^{[1]} \quad 12; C(1, 2) \equiv\equiv 12; H_2^1(2, 1) \overset{\bullet\bullet\bullet}{\bar{1}\bar{1}} \quad 12; H_2^1(2, 1) \bar{1}\bar{1} \quad 12;$$
$$(12 + 12 + 12 + 12)/48 = \mathbf{1}.$$

\mathbb{Z}_p^2 actions, $p = 3, 5, 7$, branched over 3, 4, 5, 6 points

	$p = 3$		$p = 5$		$p = 7$	
R	Genus	# Types	Genus	# Types	Genus	#Types
3	1	1	6	1	15	1
4	4	2	16	4	36	6
5	7	5	26	14	57	39
6	10	10	36	57	78	282

Table: Topological types of \mathbb{Z}_p^2 actions fully ramified over R points, $p = 3, 5, 7$.

Some references

- ▶ S.A. Broughton and A. Wootton, Finite abelian subgroups of the mapping class group, *Algebraic and Geometric Topology* **7** (2007) 1651-1697.
- ▶ A.F. Costa and S.M. Natanzon, Topological classification of \mathbb{Z}_p^m actions on surfaces, *Michigan Math. J.* **50** (2002) 451-459.
- ▶ J. Gilman, On conjugacy classes in the Teichmüller modular group, *Michigan Math. J.* **23** (1976) 53-63.
- ▶ A. Weaver, Zero-sum multisets mod p with an application to surface automorphisms (2024), ArXiv.org.

Thanks for listening!