Classifying actions of the alternating group on Riemann surfaces

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Abstract group A_{24}

Group information

Description:	A_{24}
Order:	$310\cdots000 = 2^{21} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
Exponent:	$5354228880 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
Automorphism group:	Group of order $620\cdots000 = 2^{22} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$ (generators)
Outer automorphisms:	C_2, of order 2
Composition factors:	A_{24}
Derived length:	0

This group is nonabelian and simple (hence nonsolvable, perfect, quasisimple, and almost simple).

Group statistics

Order	1	2	3	4	5
Elements	1	8792390355903	13428028220072048	2542924546378413120	1725747644222610624
Conjugacy classes	1	6	8	18	4
Divisions	1	6	8	18	4
Autjugacy classes	1	6	8	18	4

Minimal Presentations

Permutation degree:	24
Transitive degree:	24
Rank:	not computed
Inequivalent generating tuples:	not computed

Properties

Label	310224200866619719680000.a
Order	$2^{21} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
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Simple	yes
$\#G^{\mathrm{ab}}$	1
$\#Z(G)$	1
$\#\mathrm{Aut}(G)$	$2^{22} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
$\#\mathrm{Out}(G)$	2
Perm deg.	24
Trans deg.	24
Rank	not computed

Related objects

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- Supergroups
- As a transitive group

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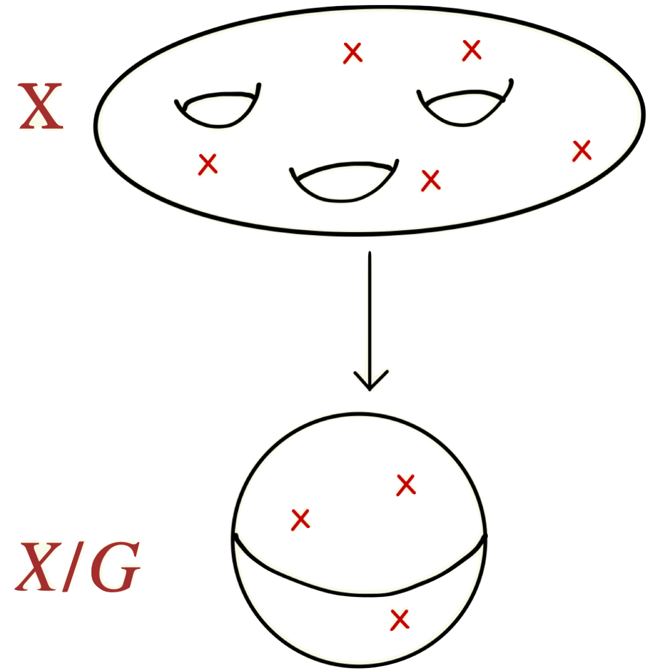
Setup

X a compact Riemann surface of genus g with $G = \mathrm{Aut}(X)$ (finite)

The set of orbits of the action is X/G, also a compact Riemann surface.



The natural map $X \to X/G$ gives us a branched covering branched at r places.

Riemann's Existence Theorem

A finite group G acts on a compact Riemann surface X of genus g >1 if and only if there are elements of the group

$$a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r$$

which generate the group, satisfy the following equation,

$$\prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} c_j = 1_G$$

$$[a, b] = aba^{-1}b^{-1}$$

h is the genus of X/G

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and so that $m_j = \mathrm{ord}(c_j)$ satisfy the Riemann Hurwitz formula

$$g = 1 + |G|(h-1) + \frac{|G|}{2} \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right).$$

signature: $[h;\ m_1, \ldots, m_r]$

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generating vector: $(a_1, b_1, \dots, a_h, b_h, c_1, \dots, c_r)$

Easy to compute.

Potential signatures are those signatures $[h; m_1, \ldots, m_r]$ which satisfy the Riemann Hurwitz formula:

$$g = 1 + |G|(h-1) + \frac{|G|}{2}\sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right).$$

Actual signatures are those which also have a generating vector associated to them.

Hard to compute.

Our Question

Which *groups* only have a finite number of potential signatures which fail to be actual signatures?

We say such groups act with almost all signatures (or are AAS).

Carvacho, P., Tucker, Wootton (2021)

Any non-abelian finite simple group is AAS.

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Any non-abelian finite simple group is AAS.

Question

Classify all actions for non-abelian finite simple groups.

$\mathcal{O}(G) = \{\mathrm{Ord}(g) : g \in G\} - \{1\}$ is the order set.

With a very small number of exceptions, any signature of the form

$$[h; \underbrace{n_1, \ldots, n_1}_{t_1}, \underbrace{n_2, \ldots, n_2}_{t_2}, \ldots, \underbrace{n_s, \ldots, n_s}_{t_s}]$$

for $n_i \in \mathcal{O}(G)$ and $t_i \in \mathbb{Z}^+$ is a potential signature.

P., Wootton

For any non-abelian finite simple group G, any potential signature $[h; m_1, \ldots, m_r]$ with $h > 1$ and the set of $m_i \in \mathcal{O}(G)$ possibly empty (i.e. no ramification) is an actual signature.

We need to find generators $a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r$ in G (may be no c_i) so that:

$$\prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} c_j = 1_G .$$

Ore Conjecture

If G is a finite non-abelian simple group, then every element of G is a commutator.

- Ore proved for A_n (1957)
- Thompson proved for PSL$(2, q)$ (1961-1962)
- Neubüser, Pahlings, Cleuvers for the sporadic simple groups (1984)
- Ellers and Gordeev (1998) and Liebeck, O'Brien, Shalev, and Tiep (2010) for simple groups of Lie type

It is well known that finite simple groups may be generated by two elements in the group

Pick a_h and b_h two generators of the group. If $r \geq 1$, also choose c_i arbitrary elements of the group of order m_i.

Since every element of the group is a commutator, $([a_h, b_h] \cdot c_1 \cdots c_r)^{-1}$ is a commutator, call it $[a_1, b_1]$. If $h > 2$ we also let $a_i = b_i = 1_G$ for $1 < i < h$.

Then $\prod_{i=1}^{h} [a_i, b_i] \prod c_i = 1_G$.

Question

Classify all actions for non-abelian finite simple groups.

Question

the alternating groups

Classify all actions for ~~non-abelian finite simple groups~~. ^

with quotient genus > 0.

Higman/Conder (1980)

The signature [0; 2, 3, 7] is the actual signature for the alternating group A_n for all but a finite number of n.

In particular all $n \geq 168$ work. There are 64 cases for $n \leq 167$ where this signature does not act.

MacBeath (1961)

The signature [0; 2, 3, 7] is the actual signature for the group PSL(2,q) if and only if q satisfies one of the following:

- $q = 7$
- $q = p$ for a prime $p \equiv \pm 1 \mod 7$
- $q = p^3$ for a prime $p \equiv \pm 2, \pm 3 \mod 7$

P., Wootton (Theorem/Conjecture)

Except for $n = 5$ or 6, every potential signature $[h;\ m_1, \ldots, m_r]$ for $h \geq 1$ is an actual signature for the group A_n.

In A_5, there is no action with signature $[1; 2]$ and in A_6 there is no action with signature $[1; 3]$.

P., Wootton (Theorem/Conjecture)

Except for $n = 5$ or 6, every potential signature $[h;\ m_1, \ldots, m_r]$ for $h \geq 1$ is an actual signature for the group A_n.

Definitely true for $n \leq 12$ and $n > 60$ish.

For $h = 1$ we need to find $c_1, \ldots, c_r$ and a, b in A_n so that $aba^{-1}b^{-1}c_1 \cdots c_r = 1_G$, and these elements generate A_n.

Two different cases:

- at least 2 branch points (periods)
- 1 branch point: $[1; k]$
- (can't have no ramification with $h = 1$)

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We describe how to choose the c_i a few slides from now.

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- There always exists a prime p with $3n/4 < p < n - 3$ for sufficiently large n. So find such prime cycles for the element $(c_1 \cdots c_r)^{-1}$. Call the prime cycles a and b_1.

- Any two p-cycles (if p is less than $n-1$) are conjugate in A_n so $b_1 = ba^{-1}b^{-1}$ and then $[a, b] \cdot c_1 \cdots c_r = 1_G$.

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A subset B of Ω is called a block if every element of G either fixes B or sends all elements of B outside of B.

A group G is primitive if it is transitive and its only blocks are the whole group, the empty set, and sets containing single elements of Ω.

A classical result attributed to Jordan says that a subgroup of S_n which is a primitive permutation group and contains a p-cycle for some prime number $p < n - 2$ must be A_n or S_n.

Miller (1928)

If a transitive group of degree n contains a cycle of prime order p for $n/2 \leq p \leq n - 3$ it must be either the alternating or symmetric group.

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For two or more periods, we show that $c_1, \ldots, c_r$ can be chosen to give a transitive action. Then by Miller's result, $a, b, c_1, \ldots, c_r$ generate all of A_n.

Two different cases:

- at least 2 branch points (periods)
- 1 branch point: $[1; m]$
- (can't have no ramification with $h = 1$)

We must find a, b, c which generate A_n with $\mathrm{ord}(c) = m$ and so that $[a, b] \cdot c = 1_{A_n}$ or $b^{-1}a^{-1}ba = c$.

A classical result attributed to Jordan says that a subgroup of S_n which is a primitive permutation group and contains a p-cycle for some prime number $p < n - 2$ must be A_n or S_n.

Jones (2013, Corollary 1.3)

Let G be a primitive permutation group of finite degree n, containing a cycle with k fixed points. Then $G \geq A_n$ if $k \geq 3$.

Bertram's result actually says that every element can be written as a product of two ℓ-cycles for any $\lceil \frac{3n}{4} \rceil \leq \ell \leq n$.

We write the element c as a product of two ℓ-cycles for $\ell = n - 3$ or $n - 4$ (depending on parity of n).

We use these ℓ-cycles (and in a few cases specify the cycle type of c) to prove primitivity.

The End