The group of automorphisms of Klein surfaces with extremal metric disks

Gou Nakamura

Aichi Institute of Technology

AMS Special Session on Automorphisms of Riemann Surfaces and Related Topics

Purpose

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   - global coordinates in Teichmüller space of genus 2,
   - algebraic equation (unfinished).

Extremal surfaces

S: a closed hyperbolic surface of genus g.

S: an extremal surface

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$\overset{\text{iff}}{\Longleftrightarrow}$ $\exists$: a regular N-gon as a fundamental region,

where $N = \begin{cases} 12g-6\text{-gon (if } S \text{ is orientable)} \\ 6g-6\text{-gon (if } S \text{ is non-orientable)} \end{cases}$

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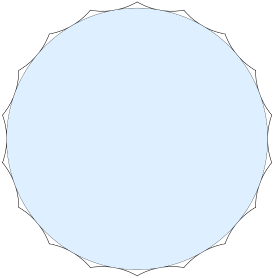


Closed Riemann surface of genus 2

$$\frac{\text{area}(D)}{\text{area}(S)} \approx 0.939693.$$

S_g: a closed Riemann surface of genus $g \geqq 2$

$$\frac{\text{area}(D_g)}{\text{area}(S_g)} = \frac{2\pi(\cosh R_g - 1)}{2\pi(2g-2)} = \frac{\frac{1}{2\sin(\pi/(12g-6))} - 1}{2g-2}$$

$$\xrightarrow{g\to\infty} \frac{3}{\pi} \approx 0.95493.$$

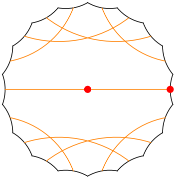
Extremal Riemann surfaces of genus 2

Theorem 1.1 (Girondo–González-Diez (2002), N (2002))

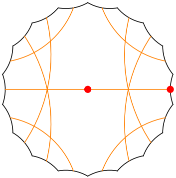
There exist exactly 9 extremal Riemann surfaces $X_1, \ldots, X_9$ *of genus 2, and all the centers of extremal disks are obtained.*

Girondo–González-Diez also obtained the Weierstrass points and the group of automorphisms for these surfaces.

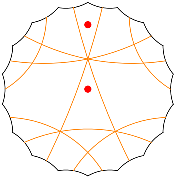
X_1



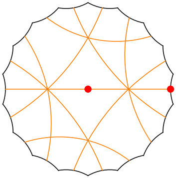
X_2



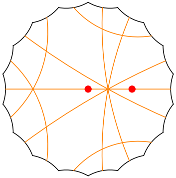
X_3



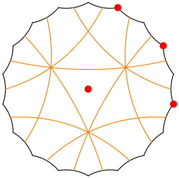
X_4



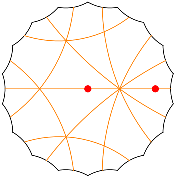
X_5



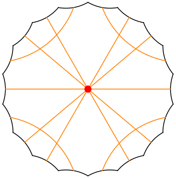
X_6



X_7



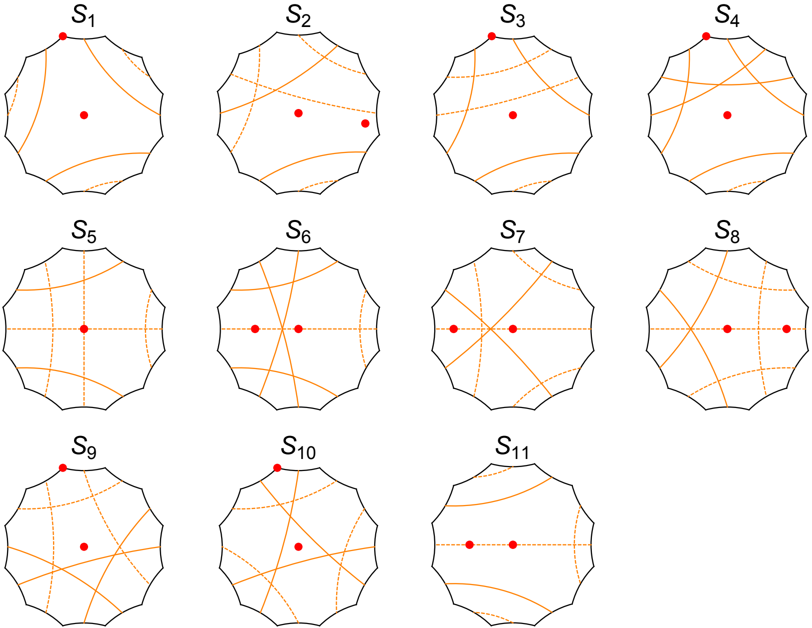
X_8



Extremal Klein surfaces of genus 3

Theorem 1.2 (Girondo-N (2007))

There exist exactly 11 extremal Klein surfaces $S_1, \ldots, S_{11}$ of genus 3, and all the centers of extremal disks are obtained.


S_1
S_2
S_3
S_4
S_5
S_6
S_7
S_8
S_9
S_{10}
S_{11}

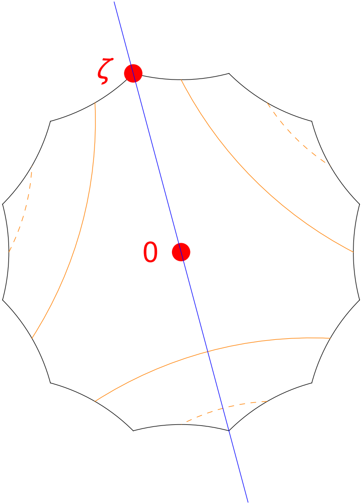
The group of automorphisms of S_1

$\mathrm{Aut}(S_1) = D_3 \times \mathbb{Z}_2 = \langle \sigma, J \rangle \times \langle JT \rangle$

$t(z) = z(\zeta - z)/(1 - \bar{\zeta} z) \rightsquigarrow T$,

$s(z) = e^{2\pi i/3} z \rightsquigarrow \sigma$.

$j(z) = e^{-5\pi i/6} \bar{z} \rightsquigarrow J$



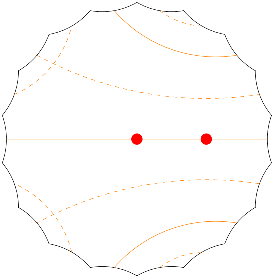
The group of automorphisms of $S_1, \ldots, S_{11}$

Surface	Aut
S_1	$D_3 \times \mathbb{Z}_2$
S_2	$\mathbb{Z}_2$
S_3	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_5	$\mathbb{Z}_2$
S_6	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_7	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_8	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_9	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_{10}	$D_3 \times \mathbb{Z}_2$
S_{11}	$\mathbb{Z}_2 \times \mathbb{Z}_2$

$g = 4$

Theorem 1.3 (N, 2009)

There exist 144 non-orientable extremal surfaces of genus 4. They admit at most two extremal discs, and 22 of them admit exactly two extremal discs.

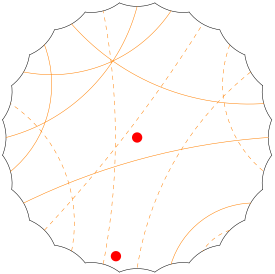


$\mathbf{Z_2} \times \mathbf{Z_2}$

$g = 5$

Theorem 1.4 (N. 2012)

There exist 3,627 non-orientable extremal surfaces of genus 5. They contain at most two extremal discs, and 17 of them contain exactly two extremal discs.

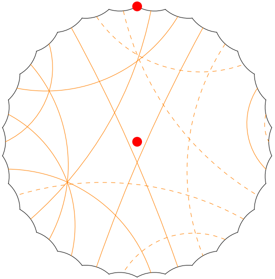


$\mathbf{Z_2}$

$g = 6$

Theorem 1.5 (N, 2016)

There exist 149,279 *non-orientable extremal surfaces of genus 6. They admit at most two extremal discs, and* 107 *of them admit exactly two.*

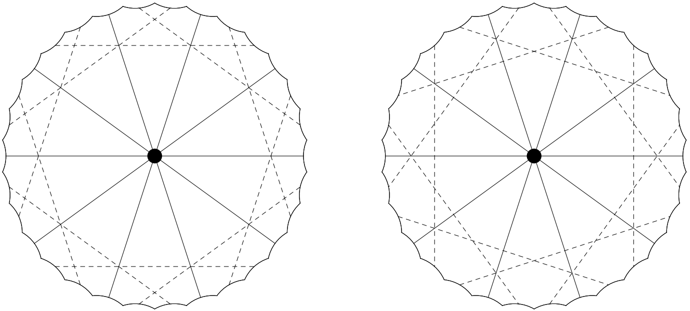


$\mathbf{Z_2}$

Theorem 1.6 (W. Hall, 1978)

The maximum order for an automorphism of a closed non-orientable surface of genus $g \geqq 3$ is $2g$ if g is odd; $2(g-1)$ if g is even.

For $g = 6$, we have two examples of order $10 = 2(g-1)$.



We can obtain sequences $\{X_{2m}\}$ of closed non-orientable extremal surfaces of genus $2m$ ($m \geq 2$). .

The complex double of a Klein surface

S: a Klein surface of algebraic genus $p \geqq 2$
$p := \#\{\text{cross caps}\} + \#\{\text{boundary components}\} - 1$

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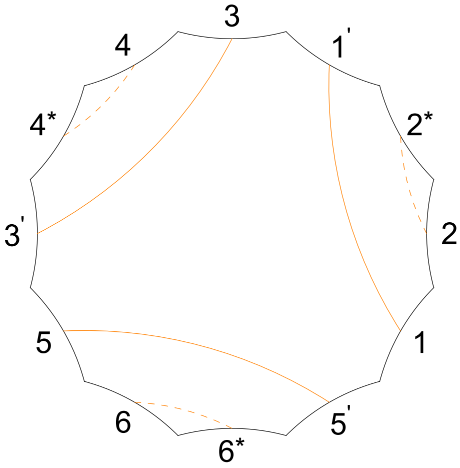
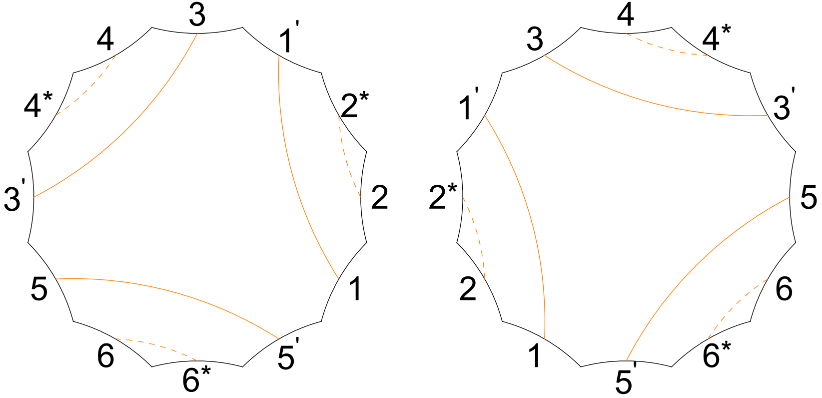
$S_C := \mathbb{D}/\Gamma^+$: the complex double of S

The complex double of S_1

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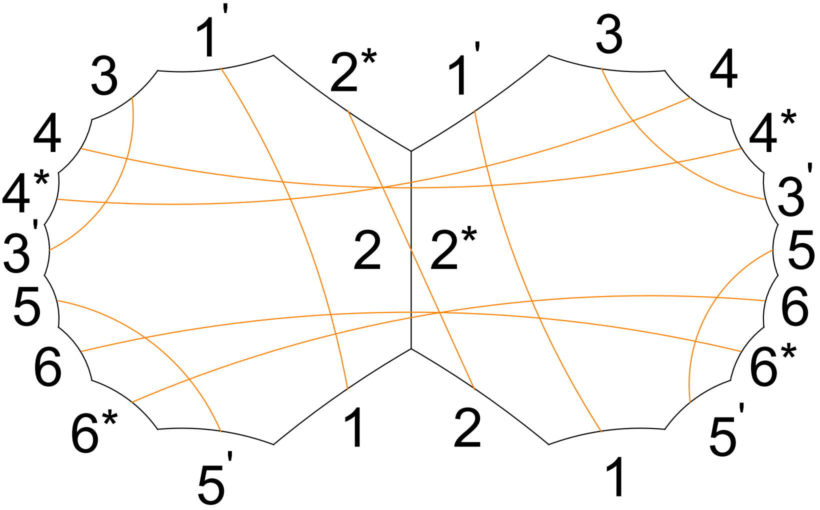
1'
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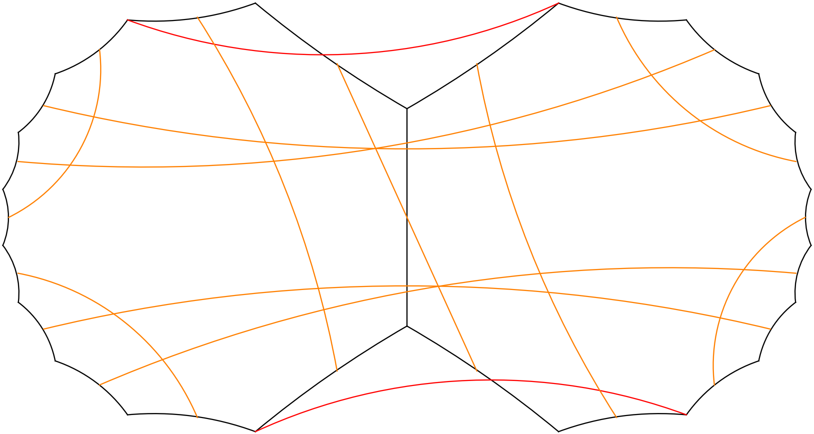
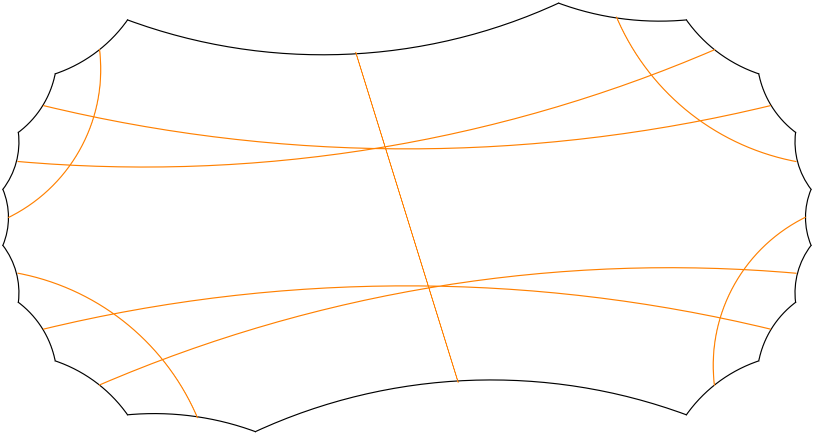
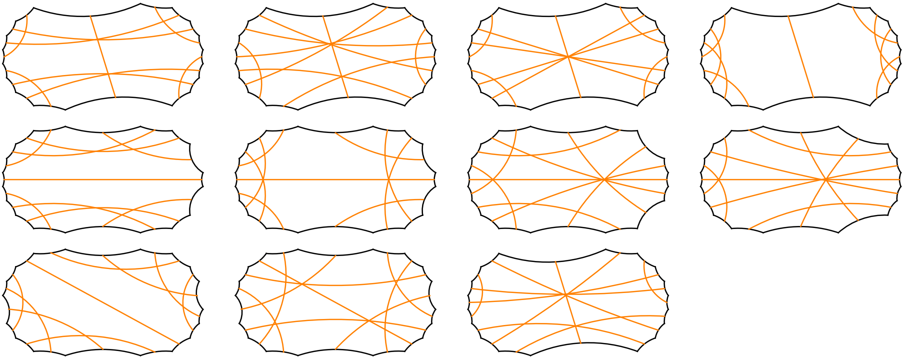


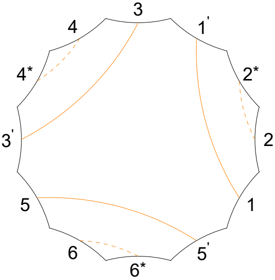


· The complex doubles $S_{1C}, \ldots, S_{11C}$



An NEC representation of S_1

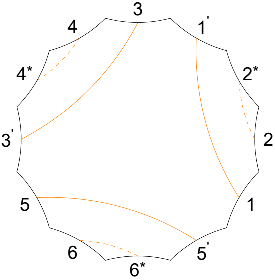
$\boldsymbol{S_1 = \mathbb{D}/\Gamma}$.



Generators of Γ: $\boldsymbol{a_1}$, $\boldsymbol{a_3}$, $\boldsymbol{a_5}$, $\boldsymbol{d_2}$, $\boldsymbol{d_4}$, $\boldsymbol{d_6}$.

$$\boldsymbol{a_1 : 1 \to 1', \quad a_3 : 3 \to 3', \quad a_5 : 5 \to 5',}$$
$$\boldsymbol{d_2 : 2 \to 2^*, \quad d_4 : 4 \to 4^*, \quad d_6 : 6 \to 6^*.}$$

$S_1 = \mathbb{D}/\Gamma$.



Generators of Γ: a_1, a_3, a_5, d_2, d_4, d_6.

$$a_1 : 1 \to 1', \quad a_3 : 3 \to 3', \quad a_5 : 5 \to 5',$$
$$d_2 : 2 \to 2^*, \quad d_4 : 4 \to 4^*, \quad d_6 : 6 \to 6^*.$$

Relations:

$$d_2^2 a_1^{-1} = 1, \quad d_4^2 a_3^{-1} = 1, \quad d_6^2 a_5^{-1} = 1, \quad a_5 a_3 a_1 = 1.$$

A Fuchsian representation of S_{1C}

$S_{1C} = \mathbb{D}/\Gamma^+$.

Generators of Γ^+.

$$\begin{array}{lllll} a_1, & a_3, & d_2^{-1}d_4, & d_2^{-1}d_4^{-1}, & a_5, \\ d_2^{-1}d_6, & d_2^{-1}d_6^{-1}, & d_2^{-1}a_5^{-1}d_2, & d_2^{-1}a_3^{-1}d_2. & \end{array}$$

A Fuchsian representation of S_{1C}

$S_{1C} = \mathbb{D}/\Gamma^+$.

Generators of Γ^+.

$$a_1,\quad a_3,\quad d_2^{-1}d_4,\quad d_2^{-1}d_4^{-1},\quad a_5,$$
$$d_2^{-1}d_6,\quad d_2^{-1}d_6^{-1},\quad d_2^{-1}a_5^{-1}d_2,\quad d_2^{-1}a_3^{-1}d_2.$$

Relations:

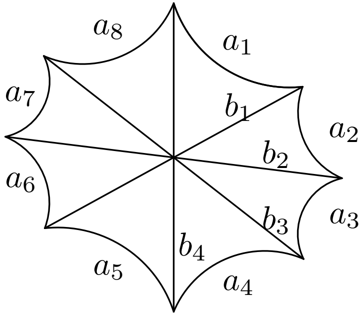
$$(d_2^{-1}a_5^{-1}d_2)(d_2^{-1}d_6)(d_6d_2) = 1,\quad (d_2^{-1}d_6^{-1})a_5(d_6^{-1}d_2) = 1,$$
$$(d_2^{-1}a_5d_2)(d_2^{-1}a_3d_2)a_1 = 1,\quad a_1^{-1}a_3^{-1}a_5^{-1} = 1,$$
$$(d_4d_2)(d_2^{-1}a_3^{-1}d_2)(d_2^{-1}d_4) = 1,\quad a_3(d_4^{-1}d_2)(d_2^{-1}d_4^{-1}) = 1.$$

Global coordinates of S_{1C}

We consider the global coodinates of S_{1C} with a certain marking in the Teichmüller space of genus two.

A model of the Teichmüller space of genus 2

Schmutz Schaller defiend Teichmüller space $\mathcal{T}_g$ of genus $g \geqq 2$ by using "canonical" polygons (1999). Following the idea, we give a model of Teichmüller space of genus 2 and consider a global coordinate system of $\mathcal{T}_2$.



$$x_1 := \cosh a_1,$$
$$x_2 := \cosh a_2,$$
$$x_3 := \cosh a_3,$$
$$x_4 := \cosh(b_1/2),$$
$$x_5 := \cosh(b_2/2),$$
$$x_6 := \cosh(b_3/2),$$
$$x_7 := \cosh(b_4/2).$$

Theorem 2.1 (N, 2012)

There exists an embedding of $\mathcal{T}_2$ *into* $\mathbb{R}^7_{>1}$*:*
$\Phi : \mathcal{T}_2 \to \mathbb{R}^7_{>1}$; $S \mapsto (x_1, \ldots, x_7)$ *satisfying the following conditions:*

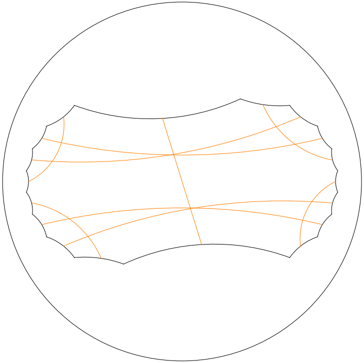
(i) $|X| < 1$, $|Y| < 1$, $|Z| < 1$;

(ii) $X + Y + Z - 1 > \sqrt{2(1-X)(1-Y)(1-Z)}$;

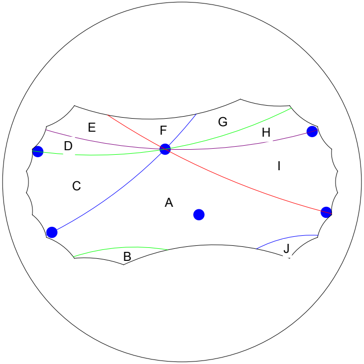
(iii) $$A^2 + B^2 + C^2 + D^2 + 2ABCD - 2 = 2\sqrt{(1-A^2)(1-B^2)(1-C^2)(1-D^2)};$$

where X, Y, Z, A, B, C, D are variables determined by $x_1, \ldots, x_7$.

The coordinates of $\tilde{S}_{1C}$ in $\mathcal{T}_2$



S_{1C}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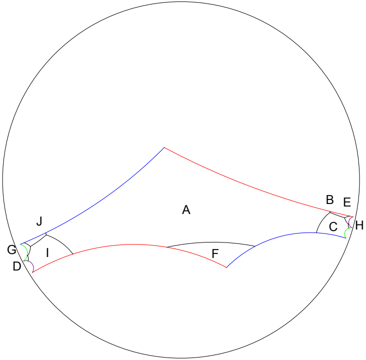
A
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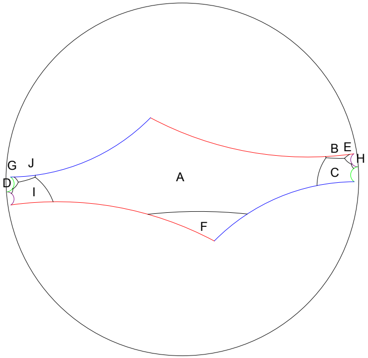
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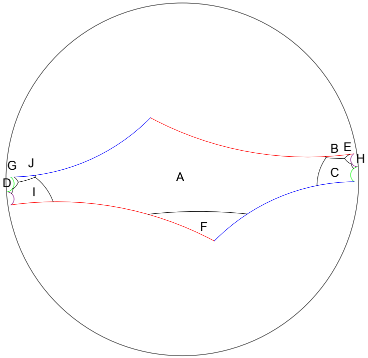

The coordinates of $\tilde{S}_{1C}$ is approximately
$(\mathbf{96.00}, \mathbf{41.28}, \mathbf{34.82}, \mathbf{57.58}, \mathbf{232.9}, \mathbf{37.55}, \mathbf{1.366})$.

We have constructed two mapping classes, a reducible one and a periodic one (to appear in Conformal Geometry and Dynamics). By their actions we will obtain different coordinates of S_{1C}.

Symmetric Riemann surfaces

A closed Riemann surface X is said to be symmetric if it admits an anticonformal involution $\sigma : X \to X$. σ is called a symmetry of X.

(X, σ) and (Y, τ) are isomorphic if there exists an conformal homeomorphism $f : X \to Y$ such that $f \circ \sigma = \tau \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \sigma \downarrow & & \downarrow \tau \\ X & \xrightarrow{f} & Y \end{array}$$

$\mathcal{M}_g^{\mathbb{R}} :=$
$\{$isomorphism classes of symmetric Riemann surfaces of genus $g\}$

The topological type of (X, σ) is the triple

$$(g, k, \varepsilon),$$

where g is the genus of X, k is the number of connected components of $\mathrm{Fix}(\sigma)$, and

$$\varepsilon := \begin{cases} 1 & \text{if } X \setminus \mathrm{Fix}(\sigma) \text{ is connected,} \\ 0 & \text{otherwise} \end{cases} .$$

$\mathcal{M}_{(g,k,\varepsilon)} :=$
$\{(X, \sigma) \in \mathcal{M}_g^{\mathbb{R}} \mid (X, \sigma) \text{ has topological type } (g, k, \varepsilon)\}$

Topological type: $g = 0, 1$

$$\mathcal{M}_0^{\mathbb{R}} = \mathcal{M}_{(0,1,0)} \cup \mathcal{M}_{(0,0,1)}$$
$$= \{(\hat{\mathbb{C}}, z \mapsto \overline{z})\} \cup \{(\hat{\mathbb{C}}, z \mapsto -1/\overline{z})\}$$

$\rightsquigarrow$ closed disk real projective plane.

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$$\mathcal{M}_1^{\mathbb{R}} = \mathcal{M}_{(1,2,0)} \cup \mathcal{M}_{(1,1,1)} \cup \mathcal{M}_{(1,0,1)}.$$

$\rightsquigarrow$ annulus Klein bottle Möbius strip

Topological type: $g = 2$

$$\mathcal{M}_2^{\mathbb{R}} = \mathcal{M}_{(2,3,0)} \cup \mathcal{M}_{(2,2,1)} \cup \mathcal{M}_{(2,1,1)} \cup \mathcal{M}_{(2,1,0)} \cup \mathcal{M}_{(2,0,1)}.$$

$\rightsquigarrow$

- spheres with 3 holes.
- projective planes with 2 holes,
- connected sums of two projective planes with a hole,
- tori with a hole,
- connected sums of 3 projective planes.

Theorem 3.1 (Weichold,1883)

The number of topological type is $\left[\frac{3g+4}{2}\right]$.

Algebraic equations of $S_{1C}, \ldots, S_{11C}$

Theorem 3.2 (Cirre, Pacific J. of Math. 208, 2003)

$$\mathcal{M}_2^{\mathbb{R}} = \mathcal{M}_{(2,3,0)} \cup \mathcal{M}_{(2,2,1)} \cup \mathcal{M}_{(2,1,1)} \cup \mathcal{M}_{(2,1,0)} \cup \mathcal{M}_{(2,0,1)}.$$

$$\mathcal{M}_{(2,3,0)} : X(a,b,c) = \{y^2 = x(x-1)(x-a)(x-b)(x-c)\},$$
$$\mathcal{M}_{(2,2,1)} : X(a,b,c) = \{y^2 = (x^2+1)(x-a)(x-b)(x-c)\},$$
$$\mathcal{M}_{(2,1,1)} : X(a,b,c) = \{y^2 = x((x-a)^2+b^2)((x-c)^2+1)\},$$
$$\mathcal{M}_{(2,1,0)} : X(a,b,c) = \{y^2 = (x^2+1)(x^2+a^2)((x-b)^2+c^2)\},$$
$$\mathcal{M}_{(2,0,1)} : X(a,b,c) = \{y^2 = -(x^2+1)(x^2+a^2)((x-b)^2+c^2)\}.$$

There are some relations between a, b, and c.

Put $\boldsymbol{\Delta} = \{0 < a < 1, b \geqq 0, c > 0,$

$$(0, a) \neq (b, c) \neq (0, 1),$$
$$(a^2 - c)(c - 1) \leqq b^2 \leqq a - c^2\}.$$

Theorem 3.3 (Cirre, 2003)

$\Psi : \Delta \to \mathcal{M}_{(2,0,1)}; (a, b, c) \mapsto X(a, b, c)$ *is a bijection, where*

$$X(a, b, c) = \{y^2 = -(x^2 + 1)(x^2 + a^2)((x - b)^2 + c^2)\}).$$

$\mathrm{Aut}(X, \sigma) := \{f : X \to X \mid$
a conformal homeomorphism such that $f \circ \sigma = \sigma \circ f\}$

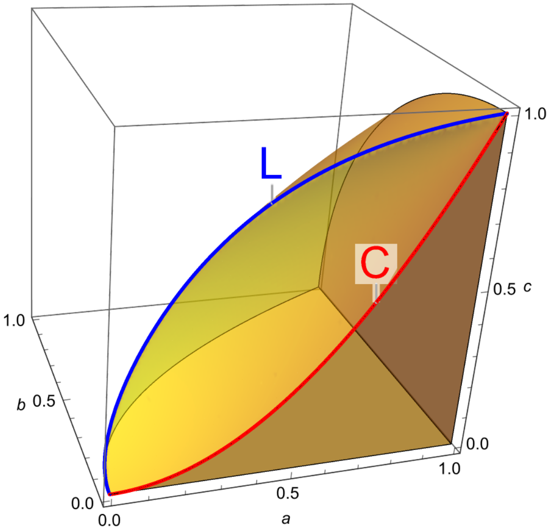
$\mathcal{M}_{(g,k,\varepsilon)}(H) := \{(X, \sigma) \in \mathcal{M}_{(g,k,\varepsilon)} \mid \mathrm{Aut}(X, \sigma) = H\}$.

Theorem 3.4 (Cirre, 2003)

$\boldsymbol{\Delta = D \cup S \cup C \cup L}$,

$$\boldsymbol{D \xrightarrow{\Psi} \mathcal{M}_{(2,0,1)}(C_2),}$$
$$\boldsymbol{S \xrightarrow{\Psi} \mathcal{M}_{(2,0,1)}(D_2),}$$
$$\boldsymbol{C \xrightarrow{\Psi} \mathcal{M}_{(2,0,1)}(D_4),}$$
$$\boldsymbol{L \xrightarrow{\Psi} \mathcal{M}_{(2,0,1)}(D_6),}$$

where $\boldsymbol{D = \{b > 0, (a^2 - c)(c - 1) < b^2 < a - c^2\}}$,
$\boldsymbol{S = \{b > 0, (a^2 - c)(c - 1) = b^2 < a - c^2\}}$
$\boldsymbol{\cup \{b > 0, (a^2 - c)(c - 1) < b^2 = a - c^2\}}$
$\boldsymbol{\cup \{b = 0, c < a^2\}}$,
$\boldsymbol{C = \{b = 0, c = a^2\}}$,
$\boldsymbol{L = \{b > 0, (a^2 - c)(c - 1) = b^2 = a - c^2\}}$


L
C
1.0
0.5 c
0.0
1.0
b 0.5
0.0
0.0
0.5
1.0
a

Since we know $\mathbf{Aut}(S_1), \ldots, \mathbf{Aut}(S_{11})$, we have the following result:

Result 3.5

- $S_{2C}, S_{5C} \in D \leftrightarrow \mathcal{M}_{(2,0,1)}(C_2)$
- $S_{3C}, S_{4C}, S_{6C}, S_{7C}, S_{8C}, S_{9C}, S_{11C} \in S \leftrightarrow \mathcal{M}_{(2,0,1)}(D_2)$
- $S_{1C}, S_{10C} \in L \leftrightarrow \mathcal{M}_{(2,0,1)}(D_6)$