Derived category of moduli space of vector bundles on a curve

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Derived category

X: smooth projective variety/$\mathbb{C}$.

The geometry of X is studied via its geometric/topological/algebraic invariants.

Cohomology ring ($\mathrm{H}^\bullet(X)$), Chow ring ($\mathrm{A}^\bullet(X)$), Motive ($h(X)$), $\cdots$

Definition.

The (bounded) **derived category** $\mathrm{D}(X)$ is a category that:
Object: finite chain complexes of coherent sheaves

$$F^\bullet : \cdots \to F^{i-1} \to F^i \to F^{i+1} \to \cdots$$

Morphism ($F^\bullet \to G^\bullet$): equivalence classes of chain maps

$$F^\bullet \xrightarrow{\phi} G^\bullet .$$

where any map induces $\mathrm{H}^i(F^\bullet) \xrightarrow{\phi\ \cong} \mathrm{H}^i(G^\bullet)$ is invertible.

Semiorthogonal decomposition

We want to decompose it into simpler subcategories.

Definition.

A **Semiorthogonal decomposition (SOD)** of $\mathrm{D}(X)$ is a sequence of subcategories $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_k$ of $\mathrm{D}(X)$ such that

1. (Independence) $\forall F_i^\bullet \in \mathcal{A}_i,\ F_j^\bullet \in \mathcal{A}_j$, $\mathrm{Hom}(F_i^\bullet, F_j^\bullet) = 0$ if $i > j$.
2. (Span) $\mathrm{D}(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_k \rangle$.

Example. (Beilinson, 78)

$$\mathrm{D}(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \cdots, \mathcal{O}(n) \rangle = \langle (n+1)\mathrm{D}(\cdot) \rangle$$

Conjecture. (Kawamata, 08)

There is a SOD on $\mathrm{D}(X)$ that is compatible with the MMP of X.

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \cdots \dashrightarrow X_m = X_{\min}$$

$\Rightarrow \mathrm{D}(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_k, \mathrm{D}(X_{\min}) \rangle$

Semiorthogonal decomposition

Then indecomposable categories are building blocks.

Example. Indecomposable derived categories

- (Okawa, 11) Curves of $g \geq 1$.
- (Kawatani-Okawa, 15) X with $|K_X|$ is base-point-free. In particular, Abelian varieties and Calabi-Yau manifolds.
- (Lin, 21) For a curve C, $\mathrm{Sym}^k C$ for $k < g(C)$.

Non-example.

- $\mathrm{D}(\mathrm{Sym}^g C) = \langle \mathrm{D}(\mathrm{Jac}(C)), \mathrm{D}(\mathrm{Sym}^{g-2} C) \rangle$.
- Fano varieties are always decomposable.

Moduli space of vector bundles

C: smooth projective curve of genus $g \geq 2$.

$L \in \mathrm{Pic}^d(C) \cdots$ line bundle of degree d on C.

Definition.

$$\mathcal{M}_C(r, L) := \{E \mid E : \mathrm{rank}\ r, \det E \cong L \text{ vector bundles on } C\}$$

$\cdots$ moduli stack of vector bundles (smooth algebraic stack).

To get a projective moduli space $\mathrm{M}_C(r, L)$, we need:

1. A vector bundle E is **stable** if for any $F \subsetneq E$, $\dfrac{\deg F}{\mathrm{rank}\ F} < \dfrac{\deg E}{\mathrm{rank}\ E}$.
   We consider the moduli space of stable bundles.
2. We assume that $(r, d) = 1$.

Theorem. (Mumford, Narasimhan-Seshadri, Ramanan, 60 – 70s)

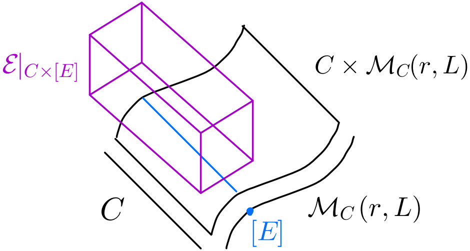
The moduli space $\mathrm{M}_C(r, L)$ of stable vector bundles is a smooth projective Fano variety of dimension $(r^2 - 1)(g - 1)$.

Moduli space of vector bundles

Definition.

The **universal bundle** $\mathcal{E}$ is a vector bundle over $C \times \mathcal{M}_C(r, L)$ such that for any point $[E] \in \mathcal{M}_C(r, L)$,

$$\mathcal{E}|_{C \times [E]} \cong E.$$



If $(r, d = \deg L) = 1$, $\exists$ a universal bundle over $C \times \mathrm{M}_C(r, L)$ (**Poincaré bundle**).

Derived category of moduli space of vector bundles

Goal.

Study a SOD of $\mathrm{M}_C(r, L)$.

Conjecture. (Gomez-Lee, 20)

The SOD of $\mathrm{D}(\mathrm{M}_C(r, L))$ can be described in terms of $\mathrm{D}(\mathrm{Sym}^n C)$ and $\mathrm{D}(\mathrm{Jac}(C))$.

Three reasons to believe:

1. Motivic computation.
2. Parametrized version ($=$ Quot scheme). (Toda, 22)
3. Rank two case.

Theorem. (Narasimhan conjecture, Tevelev-Torres, 22, 23)

$$\mathrm{D}(\mathrm{M}_C(2, L)) = \langle 2\mathrm{D}(\cdot), 2\mathrm{D}(C), \cdots, 2\mathrm{D}(\mathrm{Sym}^{g-2}C), \mathrm{D}(\mathrm{Sym}^{g-1}C)\rangle.$$

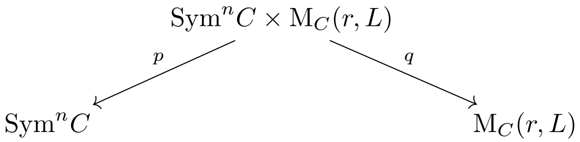
Main result

Theorem. (Lee-M, 23, 24+)

$$\mathrm{D}(\mathrm{M}_C(r,L)) = \langle \mathrm{D}(\cdot), \mathrm{D}(C), \mathrm{D}(\mathrm{Sym}^2 C), \cdots, \mathrm{D}(\mathrm{Sym}^{r-1}C), \cdots \rangle.$$

Construction (**Fourier-Mukai transform**)

1. $\mathcal{E}$: Poincaré bundle over $C \times \mathrm{M}_C(r,L)$.
2. $\pi_i : C^n \times \mathrm{M}_C(r,L) \to C \times \mathrm{M}_C(r,L)$: i-th projection.
3. $\mathcal{G} := \bigotimes_{i=1}^n \pi_i^* \mathcal{E}$: Bundle over $C^n \times \mathrm{M}_C(r,L)$ with $\mathcal{G} \curvearrowleft S_n \Rightarrow$ Descent to a bundle $\mathcal{F}$ over $\mathrm{Sym}^n C \times \mathrm{M}_C(r,L)$.



4. We can define a functor $\Phi_{\mathcal{F}} : \mathrm{D}(\mathrm{Sym}^n C) \to \mathrm{D}(\mathrm{M}_C(r,L))$ as $\Phi_{\mathcal{F}}(A^\bullet) := q_*(p^* A^\bullet \otimes \mathcal{F})$.

Key ingredients

Claim: If $r > n$, $\mathrm{D}(\mathrm{Sym}^n C) \overset{\Phi_{\mathcal{F}}}{\hookrightarrow} \mathrm{D}(\mathrm{M}_C(r, L))$.

- **Bondal-Orlov criterion**: Need to evaluate $\mathrm{H}^i(\mathrm{M}_C(r, L), S_\lambda \mathcal{E}_x^* \otimes S_\mu \mathcal{E}_y)$ where $S_\lambda \mathcal{E}_x$ is a Schur functor of $\mathcal{E}_x$.
- Littlewood-Richardson rule $\Rightarrow S_\lambda \mathcal{E}_x^* \otimes S_\mu \mathcal{E}_x \cong \bigoplus m_\nu S_\nu \mathcal{E}_x$. Need to evaluate $\mathrm{H}^i(\mathrm{M}_C(r, L), S_\nu \mathcal{E}_x)$.
- Replace the moduli space:

$$\mathrm{M}_C(r, L) \overset{\text{stack}}{\to} \mathcal{M}_C(r, L)^s \overset{\text{parabolic}}{\to} \mathcal{M}_{C,x}(r, L, \mathbf{a})$$

$$\overset{\text{gen. Hecke}}{\to} \mathcal{M}_{C,x,x'}(r, \mathcal{O}, \mathbf{a}') \overset{\text{Halpern}-\text{Leistner}}{\to} \mathcal{M}_{C,x,x'}(r, \mathcal{O})$$

- Use **Borel-Weil-Bott-Teleman** theory.

Remark/Question.

So $\mathrm{D}(\mathrm{M}_C(r, L))$ knows $\mathrm{D}(C)$ and C. (Derived Torelli holds.) Can we use it to study the automorphism of C?

Application - Fano visitor

Question. (Bondal, 11)

Let X be a smooth projective variety. Can $\mathrm{D}(X)$ be embedded into $\mathrm{D}(Y)$ for a smooth Fano variety Y? If so, X is a **Fano visitor**.

Example.

- (Narasimhan, 17) Curves.
- (Kiem-Kim-Lee-Lee, 17) Complete intersections.

From Abel-Jacobi map $\mathrm{Sym}^g C \to \mathrm{Jac}(C)$, we obtain an embedding $\mathrm{D}(\mathrm{Jac}(C)) \hookrightarrow \mathrm{D}(\mathrm{Sym}^g C) \hookrightarrow \mathrm{D}(\mathrm{M}_C(g+1, L))$ and $\mathrm{M}_C(g+1, L)$ is Fano.

Theorem. (Lee-M, 23)

1. All Jacobians are Fano visitors.
2. All principally polarized Abelian varieties of dimension ≤ 3 are Fano visitors.

Thank you!