Extremal Riemann surfaces, their properties and applications

Ewa Kozłowska-Walania

University of Gdańsk

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Preliminaries

X – compact Riemann surface;
$G = \mathrm{Aut}^{\pm}(X)$ – full automorphism group;
$G \cong \Lambda/\Gamma$ – Λ a NEC group, Γ a Fuchsian surface group;
Symmetry σ of X – an antiholomorphic involution in G.

For σ being a symmetry of a Riemann surface X, the connected component of $\mathrm{Fix}(\sigma)$ shall be called an *oval* of σ. We call σ *separating* if $X \setminus \mathrm{Fix}(\sigma)$ is disconnected, and *nonseparating* otherwise.

For a symmetry σ on a Riemann surface of genus g we define (g, k, ε) to be the *type* of σ, where k denotes the number of ovals of σ, $\varepsilon = 0$ or 1 depending on the separability of σ.

If the genus is known, we usually just write $+k$ or $-k$.

Extremal Riemann surfaces

From the studies of Bujalance, Costa, Gromadzki, Izquierdo, Natanzon two interesting classes of surfaces emerge:

s-extremal Riemann surfaces - Riemann surfaces admitting the maximal number of nonconjugate symmetries;

o-extremal Riemann surfaces (with parameter k) - Riemann surfaces admitting the maximal total number of ovals for a system of k nonconjugate symmetries (with ovals).

Questions:

1. The group structure - done;
2. The distribution of topological types of symmetries - done in o-extremal case;
3. Real equations - done in o-extremal abelian case;
4. How can we use them?

o-extremal surfaces

The bound for the total number of ovals of k nonconjugate symmetries on a Riemann surface of genus g was studied by many authors and the question has been completely answered.

For our purposes the key result is:

Theorem (Gromadzki 2000): The maximal total number of ovals of k nonconjugate symmetries on a Riemann surface of genus g does not exceed

$$2g - 2 + (9 - k)\frac{|G|}{8},$$

where G denotes the 2-group of automorphisms generated by the symmetries.

s-extremal surfaces

Let now $g = 2^{r-1}u + 1$ for some odd u.

Theorem (Bujalance, Gromadzki, Izquierdo 2001): Let $\omega(g)$ denote the maximal possible number of nonconjugate symmetries on a Riemann surface of genus g. Then:
(1) $\omega(g) \leq 2^{r+1}$;
(2) $\omega(g) = 2^{r+1} \Leftrightarrow u \geq 2^{r+1} - 3$;
(3) the remaining values of $\omega(g)$ were calculated.

Theorem (Gromadzki, EKW 2018): The automorphism group of an extremal Riemann surface is isomorphic to the direct product of a dihedral group and some number of copies of cyclic groups of order 2.

Knowing the structure of the automorphism group for extremal surfaces, we can try to revisit problems concerning the topological properties of the real nerve of the moduli space of Riemann surfaces

Nerve $\mathcal{N}(g)$ of $\mathcal{M}_g^{\mathbb{R}}$

$\mathcal{M}_g^{\mathbb{R}} = \bigcup_{(g,k,\varepsilon)} \mathcal{M}_g^{k,\varepsilon}$

$\mathcal{M}_g^{\mathbb{R}} \longmapsto \mathcal{N}(g)$ - the nerve of $\mathcal{M}_g^{\mathbb{R}}$ (Čech simplicial complex)

points in $\mathcal{N}(g) \longleftrightarrow$ strata $\mathcal{M}_g^{k,\varepsilon}$ in $\mathcal{M}_g^{\mathbb{R}}$

$(\mathcal{M}_g^{k_1,\varepsilon_1}, \ldots, \mathcal{M}_g^{k_n,\varepsilon_n})$ a simplex $\Leftrightarrow$ $\mathcal{M}_g^{k_1,\varepsilon_1} \cap \ldots \cap \mathcal{M}_g^{k_n,\varepsilon_n} \neq \emptyset$

$\Leftrightarrow$ there exists a Riemann surface X having n symmetries of types $(g, k_1, \varepsilon_1), \ldots, (g, k_n, \varepsilon_n)$.

Problem: Find the properties of $\mathcal{N}(g)$.

What is known?

Obviously, the geometrical dimension of $\mathcal{N}(g)$ is limited by the maximal number of nonconjugate symmetries.

Theorem (Gromadzki, EKW 2011-2016): Let $g = 2^{r-1}u + 1$ where u is odd. Then:

(1) If $r = 1$, which means that g is even, then $\dim_{\mathrm{G}}(\mathcal{N}(g)) = 3$;

(2) If $u \geq 2^{r-1}(2^{r+1} + 2) - 5$ then $\dim_{\mathrm{G}}(\mathcal{N}(g)) = 2^{r+1} - 1$.

We also conjectured that for odd g this is also the necessary condition on u.

But it turned out to be false. A few years later we found another configuration realizing the maximal geometrical dimension with u being strictly smaller. The question came back - what is the smallest u possible?

Problems like these usually require constructing some Riemann surfaces with specific properties. Namely, to obtain a simplex of maximal dimension one has to construct a Riemann surface admitting 2^{r+1} symmetries of distinct topological types - this requires defining a signature of some NEC group Λ and a particular epimorphism $\theta : \Lambda \to G = \mathrm{D}_{2^t} \times \mathrm{Z}_2^r$.

Usually it is enough to consider NEC group Λ, with signatures of the form

$$(0; +; [-]; (\{(2^t, 2, \overset{s-1}{\ldots}, 2)\}),$$

which is generated by consecutive canonical reflections

$$c_i, i = 0, \ldots, s, c_i^2 = 1, (c_0 c_1)^{2^t} = 1, (c_i c_{i+1})^2 = 1 (i > 0), c_0 = c_s$$

and sometimes even allowing $t = 1$, $G = \mathrm{Z}_2^{r+2}$.

Theorem (Gromadzki 1997): Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with the group G of all automorphisms of X, let $G = \Lambda/\Gamma$ for some NEC group Λ and let $\theta : \Lambda \to G$ be the canonical epimorphism. Then the number of ovals of a symmetry σ of X equals

$$\sum [C(G, \theta(c_i)) : \theta(C(\Lambda, c_i))],$$

where the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to σ.

Theorem (Singerman 1974): Let $c_0, c_1, \ldots, c_s$ be the system of canonical reflections corresponding to a period cycle $(n_1, \ldots, n_s)$ of an NEC group Λ. If all n_i are even, then the centralizer $C(\Lambda, c_i)$ equals

$$\begin{array}{ll} \langle c_i \rangle \times (\langle (c_{i-1}c_i)^{n_i/2} \rangle * \langle (c_i c_{i+1})^{n_{i+1}/2} \rangle) = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) & \text{for } i \neq 0, \\ \langle c_0 \rangle \times (\langle (c_0 c_1)^{n_1/2} \rangle * \langle (c_{s-1} c_s)^{n_s/2} \rangle) = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) & \text{for } i = 0, \end{array}$$

Minimizing u is practically equivalent to minimizing s, being the length of the period cycle as

$$\frac{g-1}{|G|/2} = \frac{u}{2^{t+1}} = \frac{\mu(\Lambda)}{2\pi} = -\frac{1}{2} - \frac{1}{2^{t+1}} + \frac{s-1}{4}.$$

In such a case the reflections of the period cycle can be seen as points situated on a circle and labeled by the symmetries in G. We allow one of the symmetries to have no ovals.

Moreover, by the centralizers theorems above, in the most interesting abelian case, a reflection contributes with $|G|/4$ ovals to its symmetry if it has neighbors with the same image under θ and it contributes $|G|/8$ ovals otherwise. This leads to the following combinatorial problem.

The combinatorial problem:

Let us consider the following problem: we have s points situated on a circle and labeled with k labels in such a way that no two consecutive points have the same label. Moreover, if a point has neighbors with the same label, then it contributes 2 to its label and if a point has neighbors with distinct labels, then it contributes 1 to its label. The weight of a label is the sum of all contributions of the points labeled with it. What is the smallest s such that each of the labels in this coloring has different weight?

Theorem (Sikorski 2022): The minimal number $\varphi(k)$ of points required for such a labeling is

$$\varphi(k) = \begin{cases} \frac{k^2+3k+2}{4}, & k \equiv 2 \text{ or } 3 \mod 4; \\ \frac{k^2+3k}{4}, & k \equiv 0 \text{ or } 1 \mod 4. \end{cases}$$

Obviously the most interesting case for us is $k = 2^{r+1} - 1$, for which $\varphi(2^{r+1} - 1) = 2^{r-1}(2^{r+1} + 1)$.

The results

Theorem: Let $g = 2^{r-1}u + 1$, odd u, Then a Riemann surface of genus g has 2^{r+1} nonconjugate symmetries such that each of the symmetries has a distinct topological type if and only if $u \geq 2^{r-1}(2^{r+1} + 1) - 3$.

Idea of proof:

1. If a Riemann surface has $2^{r+1} - 1$ symmetries with ovals, then all the symmetries are nonseparating. Therefore to have distinct topological types, we need distinct numbers of ovals, allowing one of the symmetries to have 0 ovals.
2. The maximal total number of ovals of an s-extremal surface is strictly smaller than the general upper bound, namely for $|G| = 2^{r+t+1}$ it is at most equal to

$$2^r u + (7 - 2^{r+1})2^{r+t-2} + 2^r.$$

This in particular means that s-extremal surfaces are never o-extremal.

3. The minimal contribution of a reflection is 2^{r-1}. This is also the minimal difference in number of ovals of two symmetries. In such a way we obtain some minimal number of ovals, that all the symmetries need to have together.
4. Now this amount has to be less or equal than the maximal possible number of ovals for such a g. Therefore we obtain an inequality, involving u and r and resulting with the required bound lower bound on u.
5. If we take any odd u larger than the minimal one, the construction stays the same but we compensate, for the increase in genus, in the signature with proper periods.

Some further questions that can be answered:

1. It can be shown that homology groups in dimensions 2^a are nontrivial for $g = 2^{r-1}u + 1$, $a \leq r - 1$, by constructing cycles which are not boundaries - this also requires knowledge of the automorphism group and the fact, that for large amounts k of symmetries, the maximal total number of ovals which is allowed decreases with the growth of k.

2. An s-extremal Riemann surface is never o-extremal. One can play here with these two notions by finding the maximal number of ovals for s-extremal surfaces or the maximal number of symmetries for o-extremal ones.

3. If X is an s-extremal Riemann surface admitting 2^{r+1} nonconjugate symmetries, then at most $r + 2$ of them are separating. It is also probably possible to check separability easily because of the group structure in extremal cases means that sketching Schreier coset graphs is easy.

Thank you for your attention!

NEC groups and Fuchsian groups

NEC group Λ – *discrete and cocompact* subgroup of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$. Its algebraic presentation is determined by the *signature*:

$$s(\Lambda) = (h; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_k\}),$$

where $C_i = (n_{i1}, \ldots, n_{is_i})$.
Fuchsian group Γ – an NEC group having no orientation reversing elements, hence it has signature of the form

$$s(\Gamma) = (g; +; [m_1, \ldots, m_r], \{-\}),$$

which shall be abbreviated as

$$s(\Gamma) = (g; m_1, \ldots, m_r).$$

The algebraic presentation for an NEC group Λ with signature:

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generators:	relations:
$x_i, \ i = 1, \ldots, r$	$x_i^{m_i} = 1, \ i = 1, \ldots, r$

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$c_{ij}, \; i = 1, \dots, k$, $\; j = 0, \dots, s_i$	$c_{ij}^2 = 1, \; i = 1, \dots, k$, $\; j = 0, \dots, s_i$
	$(c_{ij-1} c_{ij})^{n_{ij}} = 1, \; i = 1, \dots, k$, $\; j = 1, \dots, s_i$

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$e_i, \ i = 1, \ldots, k$	$c_{is_i} = e_i^{-1} c_{i0} e_i, \ i = 1, \ldots, k$

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$e_i, \ i = 1, \ldots, k$	$c_{is_i} = e_i^{-1} c_{i0} e_i, \ \ i = 1, \ldots, k$
$a_i, \ b_i, \ i = 1, \ldots, h$	$x_1 \ldots x_r e_1 \ldots e_k [a_1, b_1] \ldots [a_h, b_h] = 1$

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generators:	relations:
$x_i, \ i = 1, \ldots, r$	$x_i^{m_i} = 1, \ i = 1, \ldots, r$
$c_{ij}, \ i = 1, \ldots, k$; $j = 0, \ldots, s_i$	$c_{ij}^2 = 1, \quad i = 1, \ldots, k$; $j = 0, \ldots, s_i$
	$(c_{ij-1} c_{ij})^{n_{ij}} = 1, \ i = 1, \ldots, k$; $j = 1, \ldots, s_i$
$e_i, \ i = 1, \ldots, k$	$c_{is_i} = e_i^{-1} c_{i0} e_i, \ i = 1, \ldots, k$
$a_i, \ b_i, \ i = 1, \ldots, h$	$x_1 \ldots x_r e_1 \ldots e_k [a_1, b_1] \ldots [a_h, b_h] = 1$
$d_i, \ i = 1, \ldots, h$	$x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2 = 1.$

Hyperbolic area of the fundamental region of Λ is given by the formula:

$$\mu(\Lambda) = 2\pi(\varepsilon h + k - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) +$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - \frac{1}{n_{ij}})).$$

with $\varepsilon = 2$ or 1 according to the sign being $+$ or $-$.
[Hurwitz-Riemann formula] For a subgroup Λ' of finite index in an NEC group Λ we have

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda).$$

Fuchsian group Γ is called *surface* group if it is torsion free and in this case

$$s(\Gamma) = (g; -).$$

$X = \mathcal{H}/\Gamma$ is a Riemann surface of genus $g \geq 2$ and topology of X corresponds with the algebraic structure of the Fuchsian surface group Γ.

$G = \mathrm{Aut}^{\pm}(X) = \Lambda/\Gamma$ for some NEC group Λ and Fuchsian surface group Γ.

There is the *canonical* epimorphism $\theta : \Lambda \to G = \Lambda/\Gamma$