Short homology bases for hyperelliptic hyperbolic surfaces

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April 20 2024

Hyperelliptic surface S is a Riemann surface with holomorphic involution $\phi : S \to S$, and $\phi^2 = Id$

Hyperelliptic surface Can be viewed as a double cover of the Riemann sphere with $2g + 2$ simple branch points .

The *homology systole* $\mathrm{sys}_h(\cdot)$ is the length of a shortest non-separating simple closed geodesic.

For general Riemann surface $\mathrm{sys}_h(S) \leq C \log(g)$

For Hyperelliptic Riemann surface Muetzel showed $\mathrm{sys}_h(S) \leq const \approx 5.2678$

Theorem (Balacheff-Parlier-Sabourau (2012))

Let S be a hyperbolic surface of genus $g \geq 2$ with homology systole $\mathfrak{l} = \mathrm{sys}_h(S)$. Then there exist $2g$ loops $\alpha_1, \ldots, \alpha_{2g}$ which induce a basis of $H_1(S, \mathbb{Z})$ such that

$$\ell(\alpha_k) \leq C_0 \frac{\log(2g-k+2)}{2g-k+1} \cdot g,$$

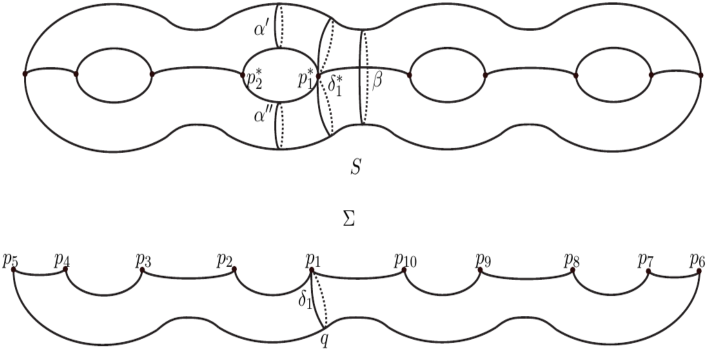
where

$$C_0 = \frac{2^{16}}{min\{\mathfrak{l}, 1\}}$$

For Hyperelliptic surfaces we will proceed in two steps

- Constructing a set of short geodesics.
- We will "prune" the set to get independent curves.

Finding short geodesics



The area on a ball around Weierstrass point p_i is
$\text{area}(B_r(p_i)) = \pi(\cosh(r) - 1)$

$$\sum_{i=1}^{2g+2} \operatorname{area}(B_{r_1}(p_i)) = (2g+2)\pi(\cosh(r_1)-1) \leq \operatorname{area}(\Sigma) = 2\pi(g-1).$$

$$\operatorname{sys}_h(S) \leq 4r_1 \leq 4\operatorname{arccosh}(2) = 5.2678...$$

Continue with this process j_m is the number of disks consumed in the steps preceding **Step m**

$$\sum_{i=j_m+1}^{2g+2} \text{area}(B_{r_m}(p_i)) \leq \text{area}(\Sigma)$$

$$\ell(\alpha_m) \leq 4\log\left(\frac{4(g-1)}{2g+2-j_m}+2\right)$$

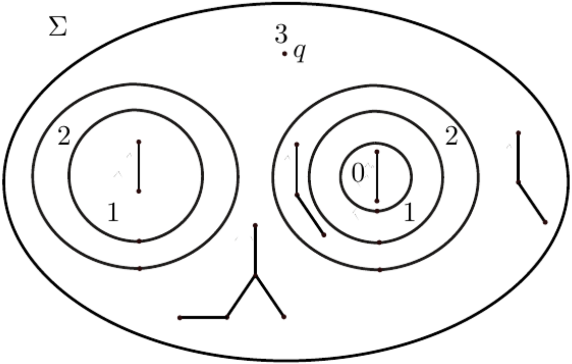
The median length ($m = \frac{g}{2}$) is bounded by $4\log(6)$ and the $(\alpha_m)_m$ are bounded by $4\log(g+1)$.

Pruning

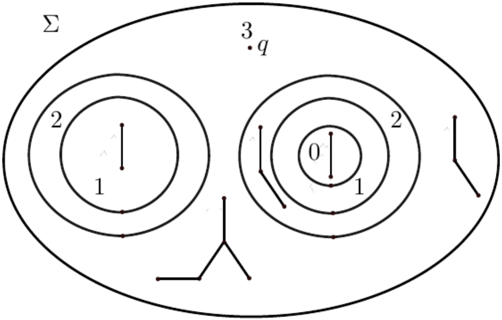
Problem the curves might be homologically dependent
i.e disconnect S.

We need to "Prune" the short curves to get a set of homologically independents curves. (set that can be extended to homology basis.)

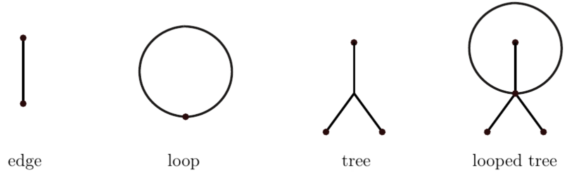
We will do so by looking at the projection $\Pi : S \to \Sigma$ of the surface to the Riemann sphere. and we look at a graph where the vertices are the Weierstrass points (fixed points of the Hyperelliptic involution) and the two (or one) vertices $\{p_i, p_j\}$ will be connected by and edge if during the process $B_r(p_i)$ touched $B_r(p_j)$



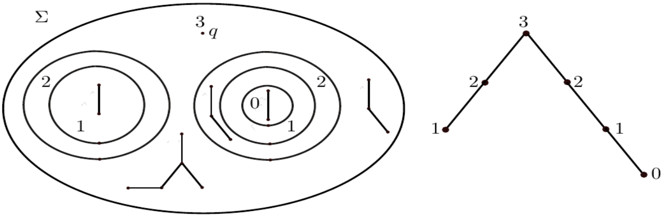
Lemma $\mathcal{H}^{\#}$ in S is **non-separating** if and only if any **open connected component of** $\Sigma \setminus \mathcal{H}$ contains a **simple closed curve** Γ that separates $p_1, \ldots, p_{2g+2}$ into two odd subsets i.e. the **number of points on either side** of Γ is **odd**.



The graph is build from the following components.



Pruning: First for every looped tree delete the loop. Second Every tree with more than one edge delete edge connected to a leaf.
Now arrange the connected regions in $\Sigma \setminus \mathcal{H}$



and at each step delete a loop and "bone"
The key part is that we can do it without loosing more then $\frac{1}{3}$ of the loops and edges. So we are left with $\frac{1}{3}(2g+2)$ independent curves

Theorem

Let S be a hyperelliptic Riemann surface of genus $g \geq 2$. Then there exist $\lceil \frac{2g+2}{3} \rceil$ geodesic loops $(\alpha_k)_{k=1,\dots,\lceil \frac{2g+2}{3} \rceil}$ that can be extended to a homology basis of $H_1(S,\mathbb{Z})$ such that

$$\ell(\alpha_k) \leq 4\log\left(\frac{12(g-1)}{2g+5-3k}+2\right) \text{ for all } k=1,\dots,\left\lceil \frac{2g+2}{3} \right\rceil.$$

Corollary

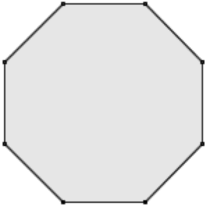
Let S be a hyperelliptic Riemann surface of genus $g \geq 2$. Then for any $\lambda \in (0,1)$ there exist $\lceil \lambda \cdot \frac{2}{3} g \rceil$ geodesic loops $(\alpha_k)_{k=1,\dots,\lceil \lambda \cdot \frac{2}{3} g \rceil}$, that can be extended to a homology basis of $H_1(S,\mathbb{Z})$ such that

$$\ell(\alpha_k) \leq N(\lambda) := 4\log\left(\frac{6}{1-\lambda}+2\right) \text{ for all } k \in \{1,\dots,\left\lceil \lambda \cdot \frac{2}{3} g \right\rceil\}.$$

Translation surfaces

a similar result can be done for Translation surfaces.
Translation surfaces can be defined in few ways I will introduced two

- A translation surface (S, ω) is a nonzero Abelian differential ω on a Riemann surface S.
- A translation surface polygon (not necessarily connected) together with a choice of pairing of parallel sides of equal length that are on "opposite sides."



As we can see the natural metric for translation surfaces is flat metric with cone point at the vertices all cone points have angle $2\pi(k+1)$
An important observation is that for embedded ball in translation surface

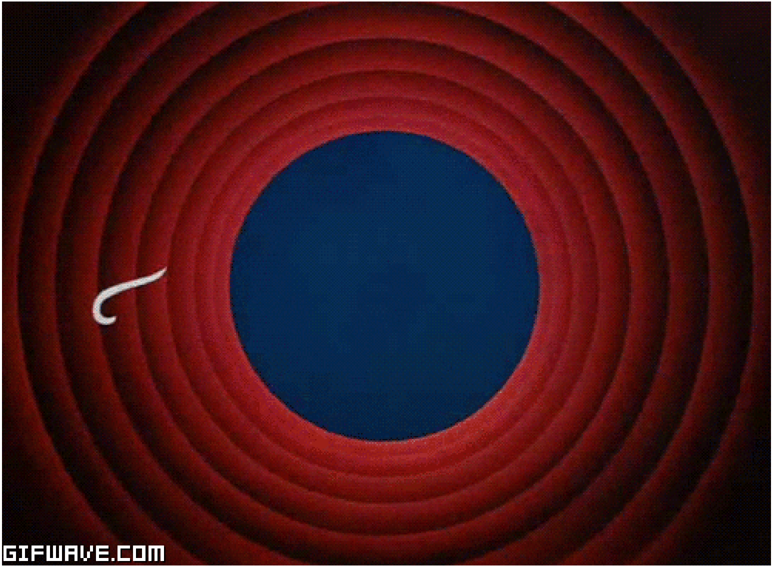
$$area(B_r(q)) \geq \pi r^2$$

We say that a translation surface is hyperelliptic is there is isometry $\phi : S \to S$ of order 2

Theorem (Hyperelliptic surfaces)

Let S be a hyperelliptic translation surface of genus $g \geq 2$ and area $4\pi g$. Then there exist $\lceil \frac{2g+2}{3} \rceil$ geodesic loops $(\alpha_k)_{k=1,\ldots,\lceil \frac{2g+2}{3} \rceil}$ that can be extended to a homology basis of $H_1(S,\mathbb{Z})$ such that

$$\ell(\alpha_k) \leq 4\cdot\sqrt{\frac{12g}{2g+5-3k}} \text{ and } \frac{\ell(\alpha_k)^2}{area(S)} \leq \frac{48}{\pi(2g+5-3k)} \text{ for all } k = 1, \ldots$$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