Geometrically Differentiating Modular Companions

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Overview - 1

- We want to geometrically compare various surfaces S of the same genus $\sigma \geq 2$ that have the same finite group G acting conformally and faithfully on them. They all have constant negative curvature and are all homeomorphic to each other.
- In general, there are infinitely many such surfaces and they are comprised of *equisymmetric strata* that form the *branch locus* in the *moduli space of genus* σ.
- Some surfaces that are “nearly the same” are called *modular companions*.
- We explain what these terms mean in the next few slides and then get to the main point of geometrically differentiating modular companions.

Overview - 2

We will discuss these ideas in sequence:

Background items

- Moduli space $\mathcal{M}_\sigma$, Teichmüller space $\mathcal{T}_\sigma$, and the modular group M_σ
- Equisymmetric strata and the branch locus
- Conformal group actions - for simplicity, planar actions only.

Main new items

- A moduli space for conformal group actions
- Modular Companions
- Equivariant tilings
- Cayley graphs
- Differentiating modular companions

Moduli space

- Moduli space $\mathcal{M}_\sigma$: space of all conformal equivalence classes of surfaces S of genus σ.
- $\mathcal{M}_\sigma$ is a quasi-projective variety of dimension $3\sigma - 3$.
- $\mathcal{M}_\sigma$ is also a complex orbifold:

$$\mathcal{M}_\sigma = \mathcal{T}_\sigma / M_\sigma. \tag{1}$$

- $\mathcal{T}_\sigma$ is Teichmüller space for genus σ.
- M_σ is the modular group for genus σ.
- $M_\sigma = MCG(S_0)$ (mapping class group) for some reference surface S_0.

Equisymmetry - 1

We now define equisymmetry.

- $\mathrm{Aut}(S)$ determines a finite subgroup $F \subset M_\sigma$, isomorphic to $\mathrm{Aut}(S)$.
- F is not unique, but its conjugacy class in M_σ, is unique.
- The conjugacy class, denoted $\Sigma(S)$, is called the *symmetry type* of S.
- Symmetry type is a conformal invariant.
- The symmetry type determines an equisymmetry relation on the moduli space: Two surfaces S_1 and S_2 are *equisymmetric* if

$$\Sigma(S_1) = \Sigma(S_2).$$

Equisymmetry - 2

- The moduli space is a disjoint union of equisymmetry equivalence classes called *equisymmetric strata*.
- The strata are smooth, irreducible quasi-projective subvarieties of $\mathcal{M}_\sigma$. Hence, they are connected.
- For $\sigma \geq 2$, $\mathrm{Aut}(S)$ is finite with order at most $84(\sigma - 1)$.
- The number of groups and hence the number of strata is also finite. (use Riemann Existence Theorem)
- There are about 26 strata in genus 2 and about 56 or so in genus 3.

Teichmuller space and modular group

- Teichmüller space $\mathcal{T}_\sigma$: Conformal equivalence classes marked Riemann surfaces.
- $\mathcal{T}_\sigma$ is an open cell in $\mathbb{C}^{3\sigma-3}$ with a complicated boundary.
- There are at least two equivalent definitions of $\mathcal{T}_\sigma$.
- Modular group $M_\sigma \simeq MCG(S_0)$: various definitions M_σ, tied to definitions of $\mathcal{T}_\sigma$.
- M_σ acts upon $\mathcal{T}_\sigma$ as a group of biholomorphic transformations, with a properly discontinuous action, and

$$\mathcal{M}_\sigma = \mathcal{T}_\sigma / M_\sigma.$$

Branch locus

- Image of fixed point subsets of $\mathcal{T}_\sigma$ is the branch locus.
- Branch locus $\mathcal{B}_\sigma$: The set of surfaces that have "higher than expected symmetry".
- $\mathcal{M}_\sigma - \mathcal{B}_\sigma$ is the set of surfaces with generic symmetry.
- $\mathcal{B}_\sigma$ is a disjoint union of equisymmetric strata.
- An ultimate goal is to describe the topology of equisymmetric strata.
- Considerable work on the branch locus (especially super-elliptic curves) has been done by M. Izquierdo, A. Costa, G. Bartolini, H. Parlier, as well as others.

Start with the automorphism group! - 1

To find a surface with a finite group G of automorphisms, we do the following:

- find a Fuchsian group Γ (*covering group*) with a nice generating system;
- determine a suitable, finite index, normal, surface group (no elliptic elements.i.e., torsion free)

$$\Pi \lhd \Gamma;$$

- then, the finite group $G \simeq \Gamma/\Pi$ acts naturally as a group of automorphisms on the smooth surface $S \simeq \mathbb{H}/\Pi$.

Start with the automorphism group! - 2

In the other direction, given G, we do the following:

- Search for Γ and epimorphisms $\eta : \Gamma \to G$ that preserve the order of elliptic elements - called *surface kernel epimorphisms*.
- For each such η, $\Pi = \ker(\eta)$ is the desired surface group, and $G \simeq \Gamma/\Pi$ acts as a group of automorphisms of $S \simeq \mathbb{H}/\Pi$.
- The inverse map $\epsilon : G \hookrightarrow \mathrm{Aut}(S)$ of $\overline{\eta} : \Gamma/\Pi \leftrightarrow G$ is the *conformal action* of G on S determined by η.

Start with the automorphism group! - 3

- For simplicity, assume that $\mathbb{H}/\Gamma = S/G$ is a topological sphere, a so-called *planar action*.
- Then Γ has a presentation:

$$\Gamma = \left\langle \gamma_1, \ldots, \gamma_r : \prod_{i=1}^{r} \gamma_i = \gamma_1^{n_1} = \cdots = \gamma_r^{n_r} = 1 \right\rangle$$

- Let $c_i = \eta(\gamma_i)$. The r-tuple $(c_1, \ldots, c_r)$ satisfies:

$$\begin{aligned} G &= \langle c_1, \ldots, c_r \rangle, \\ o(c_i) &= n_i, \\ c_1 \cdots c_r &= 1. \end{aligned}$$

- $(c_1, \ldots, c_r)$ is called a *generating* $(n_1, \ldots, n_r)$*-vector* of G.

Start with the automorphism group! - 4

- The c_i act as rotations of S of order n_i with one or more fixed points.
- The r-tuple $\mathfrak{s} = (n_1, \ldots, n_r)$ is called the *signature* of the action, it is unique up to order.
- Surface kernel epimorphisms $\eta : \Gamma \to G$ are in 1-1 correspondence with generating vectors.
- Generating vectors are easy to classify. So, we have completely solved the problem of finding automorphism groups, but we have not yet solved the moduli problem of surfaces with automorphisms.

Now start with the orbifold quotient! - 1

It is convenient to use the terminology of complex orbifolds.

- The quotient $T = S/G = \mathbb{H}/\Gamma$ is a hyperbolic, complex, orbifold sphere with r cone points $z_1, \ldots, z_r$ of order $n_1, \ldots, n_r$, respectively.
- The orbifold fundamental group $\pi_1^{orb}(T)$ acts upon $\mathbb{H}$ via an isomorphism $\Lambda : \pi_1^{orb}(T) \leftrightarrow \Gamma$, defined by path lifting.
- The map $\eta \leftrightarrow \eta \circ \Lambda = \xi$ is a bijection of surface kernel epimorphisms to smooth monodromies

$$\xi : \pi_1^{orb}(T) \twoheadrightarrow G,$$

  (epimorphisms with torsion free kernel).
- Similar remarks on the previous slides apply to the relation between smooth monodromies and generating vectors.

Now start with the orbifold quotient! - 2

- A conformal action of G upon S determines a pair (T, ξ) where $T = S/G$ and $\xi : \pi_1^{orb}(T) \twoheadrightarrow G$ is the associated smooth monodromy.
- Conversely, given such a pair (T, ξ) there is a G action on $S \simeq \mathbb{H}/\ker \xi$.
- Let $\omega \in \mathrm{Aut}(G)$. If $\xi' = \omega \circ \xi$ then (T, ξ') yields the same surface S with twisted action $\epsilon \circ \omega^{-1}$.
- We may construct a moduli space for actions as an orbifold covering space via

$$(T, \xi^{aG}) \to T,$$

where $\xi^{aG} = \{\omega \circ \xi : \omega \in \mathrm{Aut}(G)\}$ is a *smooth monodromy class*.

Now start with the orbifold quotient! - 3

- Fix the signature $\mathfrak{s}$ of the G action or alternatively the "signature" of the quotient.
- A Teichmüller space, $\mathcal{T}_{\mathfrak{s}}$, a modular group $M_{\mathfrak{s}}$, and moduli space $\mathcal{M}_{\mathfrak{s}} = \mathcal{T}_{\mathfrak{s}}/M_{\mathfrak{s}}$. may be defined to analyze the conformal equivalence classes of quotients.
- The modular group $M_{\mathfrak{s}}$ may be realized as $\mathrm{Out}^+(\pi_1^{orb}(T_0))$, the orientation preserving outer automorphisms of $\pi_1^{orb}(T_0)$, for a reference quotient surface T_0.
- $\psi \in M_{\mathfrak{s}}$ acts (on the left) upon the finitely many smooth monodromy classes via

$$\xi^{aG} \to (\xi \circ \psi^{-1})^{aG}$$

A moduli space for actions

Theorem

Fix a signature $\mathfrak{s}$. Let $\mathcal{T}_{\mathfrak{s}}$, $M_{\mathfrak{s}}$, $\mathcal{M}_{\mathfrak{s}} = \mathcal{T}_{\mathfrak{s}}/M_{\mathfrak{s}}$ be as previously described. Let $\mathcal{F}_{\mathfrak{s}}$ be the set of smooth monodromy classes, also as previously described. Then we have:

- $\mathcal{T}_{\mathfrak{s}} \times \mathcal{F}_{\mathfrak{s}}$ *is a smooth complex manifold.*
- $M_{\mathfrak{s}}$ *acts upon* $\mathcal{T}_{\mathfrak{s}} \times \mathcal{F}_{\mathfrak{s}}$ *by a diagonal action.*
- *The map*
$$(\mathcal{T}_{\mathfrak{s}} \times \mathcal{F}_{\mathfrak{s}})/M_{\mathfrak{s}} \to \mathcal{M}_{\mathfrak{s}} = \mathcal{T}_{\mathfrak{s}}/M_{\mathfrak{s}},$$
induced by $(T, \xi^{aG}) \to T$, *is an orbifold covering space of the space of quotients* $\mathcal{M}_{\mathfrak{s}}$.
- *The components of* $(\mathcal{T}_{\mathfrak{s}} \times \mathcal{F}_{\mathfrak{s}})/M_{\mathfrak{s}}$, *called strata, are in 1-1 correspondence with the orbits of* $M_{\mathfrak{s}}$ *acting upon* $\mathcal{F}_{\mathfrak{s}}$.

Modular companions

Fix a finite group and a signature $\mathfrak{s}$. Per the previous theorem, set

$$\mathcal{S}_{G,\mathfrak{s}} = (\mathcal{T}_{\mathfrak{s}} \times \mathcal{F}_{\mathfrak{s}})/M_{\mathfrak{s}}$$

with covering map

$$q : \mathcal{S}_{G,\mathfrak{s}} \to \mathcal{M}_{\mathfrak{s}}.$$

Definition

Two surfaces S_1, S_2 with $G, \mathfrak{s}$ action are modular companions if S_1/G and S_2/G are conformally equivalent, $(q(S_1) = q(S_2))$, and S_1 and S_2 correspond to point in the same path component of $\mathcal{S}_{G,\mathfrak{s}}$.

Alternatively, there is a one parameter family of surfaces with G action such that S_1 and S_2 are the beginning and ending surfaces, respectively.

Group action strata: Examples

In the table below we give information for the covering map:

$$q : \mathcal{S}_{G,\mathfrak{s}} \to \mathcal{M}_{\mathfrak{s}}.$$

- 3rd column: signature, 4th column: genus of S, 5'th column total #surfaces lying over generic S/G.
- 6'th column is a list of #Modular Companions when restricted to a component (action stratum) of the cover q.

G	$\lvert G\rvert$	$\mathfrak{s}$	σ	#surfs	#ModComp
$Sym(3)$	6	$(2,2,3,3)$	2	2	$(1,1)$
$Alt(5)$	60	$(2,2,3,3)$	11	9	(9)
$Alt(5)$	60	$(2,3,3,5)$	20	20	(20)
$Alt(5)$	60	$(5,5,5,5)$	37	47	$(6,10,15,16)$
$PSL(2,7)$	168	$(2,2,3,3)$	29	15	(15)
$PSL(2,7)$	168	$(7,7,7,7)$	121	95	$(6,7,16,24,42)$
$PSL(2,11)$	660	$(5,5,5,5)$	397	4906	not done

Equivariant tilings - setup

From a geometric perspective, a

$$\text{“smooth mondromy } \xi : \pi_1^{orb}(T) \to G\text{”}$$

is a somewhat unsatisfying classifier, so we use a “puzzle piece” construction.

To recap we have:

- A quotient map
$$\pi_G : S \to S/G = \widehat{\mathbb{C}}.$$
- The map is branched over the cone points $z_1, \ldots, z_r \in \widehat{\mathbb{C}}$.
- Over the cone point z_j, the quotient map has the local form $z \to z^{n_j}$.

Equivariant tilings - cut systems -1 (cut to pictures first)

Select a suitable cut system (embedded graph, see next slide) $\mathcal{E} \subset \widehat{\mathbb{C}}$ such that:

- The vertices of $\mathcal{E}$ contain all the cone points and perhaps one regular point.
- The arcs of $\mathcal{E}$ are smooth, meet only at vertices, and have definite, distinct tangents at the vertices.
- The complement $\widehat{\mathbb{C}} - \mathcal{E}$ is an open polygon.
- diagrams of the some sample cut systems and the corresponding polygon models are shown on the next two slides.

Equivariant tilings - cut systems - 2

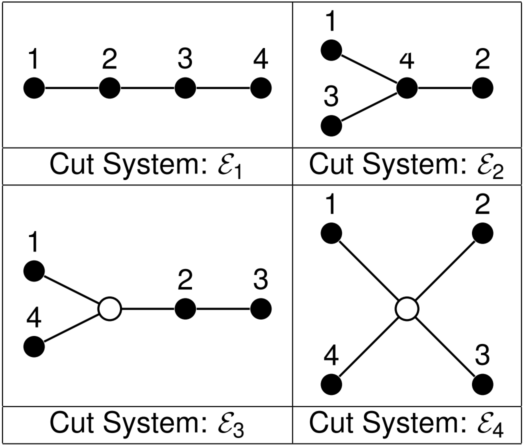


Figure 1: Various cut systems $\mathcal{E}_i$

Equivariant tilings - polygons - 1

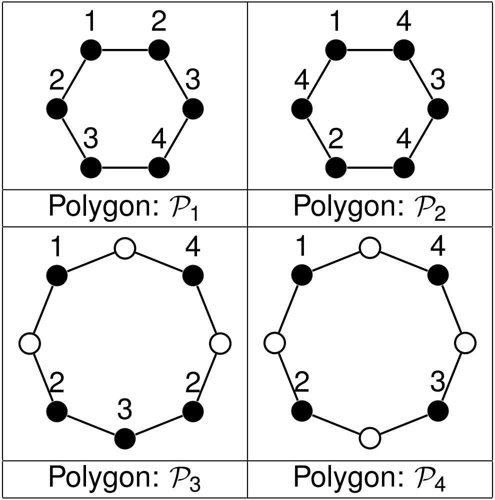


Figure 2: Various model polygons $\mathcal{P}_i$

Equivariant tilings - polygons - 2

Using the branched cover π_G, lift up the cut system $\mathcal{E}$ to an embedded graph $\tilde{\mathcal{E}}$ on S.

Polygons (puzzle pieces):

- $S - \tilde{\mathcal{E}}$ is a disjoint union of open polygons, each of which is conformally equivalent to $\widehat{\mathbb{C}} - \mathcal{E}$, and upon which G acts simply transitively.
- Select a distinguished polygon $\mathcal{P}^\circ$, and label the open polygons via

$$g \leftrightarrow g\mathcal{P}^\circ.$$

Edges:

- In $\mathcal{E}$, set $k = \#arcs = \#nodes - 1$. In S, the open polygon $\mathcal{P}^\circ$, has a boundary of $2k$ edges, $e_1, \ldots, e_{2k}$ in counter-clockwise order.
- Each edge e_i has an oppositely oriented edge $e_j = e_i^{op}$

Equivariant tilings - polygons - 3

Edges (continued):

- Each open polygon $g\mathcal{P}^\circ$ in $S - \tilde{\mathcal{E}}$ has a boundary of oriented edges of the same type as $\partial\mathcal{P}^\circ$, via

$$e_i \leftrightarrow ge_i.$$

Vertices:

- Over a white node angles are preserved.
- Over the node z_j the angles are divided by n_j (look at example).

Equivariant tilings - Rulebook

- The open polygon $\mathcal{P}^\circ$ meets a unique polygon $\tau_{e_i}\mathcal{P}^\circ$ along the edge e_i of $\mathcal{P}^\circ$.
- **Rulebook:** The following is easily proven. For $g \in G$

$$g\mathcal{P}^\circ \text{ meets } (g\tau_{e_i})\mathcal{P}^\circ \text{ along } ge_i \subset \partial g\mathcal{P}^\circ \tag{2}$$

- τ_{e_i} is called a crossover transform, and

$$\tau_{e_i^{op}} = \tau_{e_i}^{-1}$$

- The crossover transformations are easily computed from the generating vector.

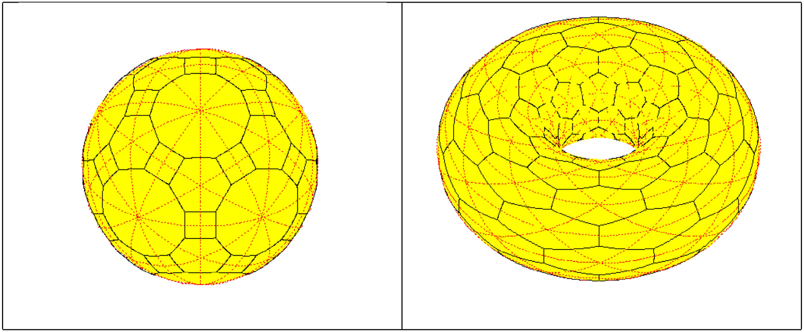
Cayley Graphs - 1 (Pictures first)

Embedded Cayley graph (geometrical): (See figures on next slide)

- Vertices: Pick $v_0 \in \mathcal{P}^\circ$. The vertices are $\{gv_0 : g \in G\}$.
- Edges: For each e_i and $g \in G$, pick an arc in S from gv_0 to $(g\tau e_i)v_0$, that crosses ge_i transversely.
- The first portion lies in $g\mathcal{P}^\circ$, crosses ge_i transversally in a single interior point, and the second portion lies in $g\tau_{e_i}\mathcal{P}^\circ$
- This can be done G equivariantly.

Cayley Graphs - 2

Embedded Cayley graph examples:



Cayley Graphs - 3 (lightly)

Abstract Cayley graph (group theory):

- Vertices: The elements of G.
- Edges: for each e_i and $g \in G$, construct the edge $(e_i, g, g\tau_{e_i})$ = (edge crossed, start polygon, ending polygon)
- Slightly different than standard definition - we can have loops and multiple arcs between vertices.
- G acts on the left: for $h \in G$, vertices $g \to hg$, edges $(e_i, g, g\tau_{e_i}) \to (e_i, hg, hg\tau_{e_i})$.
- If desired, we may colour the edges with a different colour for each edge type e_i.

Partial Isometries - 1

- Let S, S' be two surfaces lying over the same quotient T, $\pi_G : S \to T$ and $\pi'_G : S' \to T$
- The map
$$\phi_0 = (\pi'_G)^{-1} \circ \pi_G : \mathcal{P}^\circ \to (\mathcal{P}^\circ)'$$
is an isometry of the distinguished open polygons.
- For $g, g' \in G$
$$g'\phi_0 g^{-1} : g\mathcal{P}^\circ \to g\prime(\mathcal{P}^\circ)'$$
is an isometry.

Partial Isometries - 2

Assume/Define

- $H \subseteq G$ is a subset containing the identity
- $w : H \leftrightarrow H' \subseteq G, h \to h'$ is some bijective mapping preserving the identity.
- Define

$$H\mathcal{P}^\circ = \bigcup_{h \in H} h\mathcal{P}^\circ, \;\; H'(P^\circ)' = \bigcup_{h' \in H'} h'(P^\circ)'$$

Then

$$\phi_H = \bigcup_{h \in H} h'\phi_0 h^{-1} : H\mathcal{P}^\circ \to H'(P^\circ)'$$

is a conformal bijection of **disconnected** sets.

Partial Isometries - 3

Assume

- $(\overline{H\mathcal{P}^\circ})^\circ$, the interior of the closure of $H\mathcal{P}^\circ$, is connected.
- the map ϕ_H has a continuous extension to $(\overline{H\mathcal{P}^\circ})^\circ$
- H is maximal with respect to the above properties

Then

- The extension of ϕ_H is called a *maximal, partial isometry*
- Such extensions are not unique, even the number of polygons is not unique.
- Maximal partial isometries may be constructed by extending ϕ_0, one polygon at a time.

Partial Isometries - 4

Partial abstract Cayley graphs may be used analyse partial isometries. We have:

Theorem

Suppose that S and S' are modular companions that both carry equivariant tilings, obtained by lifting the same cut system on their common quotient $S/G = S'/G$. Then, the two surfaces are conformally equivalent (respecting the tilings) if and only if the abstract Cayley graphs are isomorphic, respecting edge types.

Partial Isometries - 5

- Some calculations have been performed using various algorithms to construct partial isometries.
- The number of tiles in a maximal partial isometry is used as a crude measure of differences between modular companions.
- The algorithms always yield $|G|$ tiles for conformally equivalent surfaces.
- If the companions are not conformally equivalent then the number of tiles can vary but is always much less than $|G|$.
- show matrix - separate file

References

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Questions

- Any questions?