Automorphism groups of rational functions

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Rational functions I

Let k be a field and $\phi \in k(x)$, say $\phi(x) = \frac{\mathbf{F}(x)}{\mathbf{G}(x)}$ for $\mathbf{F}, \mathbf{G} \in k[x]$. A point $\mathbb{P}^1$ is called a **fixed point** for ϕ if $\phi(= .Hence$, is a fixed point if it is a root of the following polynomial

$$S(x) := \mathbf{F}(x) - x\, \mathbf{G}(x),$$

which is at most a degree $(d+1)$ equation in x. Hence, a degree d rational function has at most $(d+1)$ fixed points. We denote the set of fixed point of ϕ by $\mathbf{Fix}(\phi)$. Notice that if the set

$$\mathbf{Fix}(\phi) = \{w_1, \ldots, w_{d+1}\}$$

is known, then we can uniquely determine the rational function ϕ by solving the linear system

$$\mathbf{F}(w_i) - w_i\, \mathbf{G}(w_i) = 0,$$

for $i = 1, \ldots, d+1$ in terms of the coefficients of $\mathbf{F}$ and $\mathbf{G}$. A function ϕ has less than $d+1$ fixed points exactly when the discriminant $\Delta(S, x)$ vanishes.

An **automorphism** of ϕ is called an automorphism of $k(\phi)/k$. Hence, any $\sigma \in \text{Aut}(k(x)/k)$ such that $\phi \circ \sigma = \sigma \circ \phi$. In other words, any $\sigma \in \text{PGL}_2(k)$ such that $\phi \circ \sigma = \sigma \circ \phi$. The set of automorphisms of ϕ is $\text{Aut}(k(\phi)/k)$ and isomorphic to

$$\text{Aut}(\phi) := \{\sigma \in \text{PGL}_2(k) \,:\, \sigma \circ \phi \circ \sigma^{-1} = \phi\}$$

Lemma
$Aut(\phi)$ acts on $\mathbf{Fix}(\phi)$ by permutation. Moreover, if $p \in \mathbf{Fix}(\sigma)$ and $\sigma \in Aut(\phi)$ then $\phi(p) \in \mathbf{Fix}(\sigma)$.

Rational functions II

Thus, $Aut(\phi)$ is finite since it permutes the finite set of fixed points of $\phi(x)$. Since $\langle\sigma\rangle$ has no fixed points in $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$, then $|\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)| = |\sigma| \cdot \#orbits$. Hence, if $\sigma \in Aut(\phi)$ then $|m|$ divides the cardinality of $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$.

Proposition

Let $G := Aut(\phi)$, $\sigma \in G$, such that $|\sigma| = m$. Then $H := \langle\sigma\rangle$ acts on $\phi^{-1}(0)$ and $\phi^{-1}(\infty)$. Hence, $\phi(x)$ can be written as

$$\phi(x) = x\,\psi(x^m),$$

where $\psi(x)$ is a rational function. Moreover, for $G \cong A_4, S_4, A_5$ then $m = 2, 4, 5$.

Let $\sigma \in PGL_2(k)$, such that $\sigma = \begin{pmatrix} a & b \\ c & e \end{pmatrix}$. For $\mathbf{F}(x) \in k[x]$, denote by

$$\mathbf{F}^\sigma(x) := \mathbf{F}(ax + b, cx + d). \tag{1}$$

Two rational functions $f, g \in \mathcal{F}_d$ are called k-**equivalent** if and only if there exists a matrix $\sigma \in PGL_2(k)$ such that $g = \sigma^{-1} \circ f \circ \sigma$. We denote by

$$f^\sigma := \sigma^{-1} \circ f \circ \sigma \tag{2}$$

Notice two different uses of notation f^σ for rational functions and $\mathbf{F}^\sigma$ for binary forms. The function $\phi^\sigma = \sigma^{-1}\phi\sigma$ is explicitly given as

$$(\sigma^{-1}\phi\sigma)(x) = \frac{e\,\mathbf{F}(ax+b, cx+e) - b\,\mathbf{G}(ax+b, cx+e)}{-c\,\mathbf{F}(ax+b, cx+e) + a\,\mathbf{G}(ax+b, cx+e)} = \frac{e\,\mathbf{F}^\sigma - b\,\mathbf{G}^\sigma}{-c\,\mathbf{F}^\sigma + a\,\mathbf{G}^\sigma} \tag{3}$$

Rational functions III

Let $G := \text{Aut}(\phi)$, $\sigma \in G$, such that $|\sigma| = m$. Then $H := \langle \sigma$ acts on $\phi^{-1}(0)$ and $\phi^{-1}(\infty)$. Hence, $\phi(x)$ can be written as

$$\phi(x) = x\,\psi(x^m),$$

where $\psi(x)$ is a rational function. Moreover, for $G \cong A_4, S_4, A_5$ then $m = 2, 4, 5$.

Lemma

Let $G := \text{Aut}(\phi)$, $\sigma \in G$, such that $|\sigma| = m$ and $\phi(x) = x\,\frac{F(x^m)}{G(x^m)}$ as in ??. Then $\tau(x) = \frac{1}{x}$ is an automorphism of ϕ (i.e. $\tau \in G$) if and only if

$$\phi(x) = x\,\frac{x^{sm} + a_{s-1}x^{(s-1)m} + \ldots + a_1 x^m + 1}{x^{sm} + a_1 x^{(s-1)m} + \ldots + a_{s-1}x^m + 1}$$

Rational functions IV

Theorem
Let $\phi(x) \in k(x)$, deg $\phi = d \geq 2$, and $G = Aut(\phi)$. Then one of the following occurs:

1. $C_m \hookrightarrow G$: Let

$$h(x) = \frac{\Lambda(a_1, \ldots, a_r, x)}{\Lambda(b_1, \ldots, b_r, x)} = \frac{x^{rm} + a_{r-1}x^{(r-1)m} + \cdots a_1 x^m + 1}{x^{rm} + b_{r-1}x^{(r-1)m} + \cdots b_1 x^m + 1}$$

$C_m \hookrightarrow G$ if and only if one of the following cases occurs: i) $m \mid (d - 1)$, $r = \frac{d-1}{m}$ and $\phi(x) = x\,h(x)$. ii) $m \mid d$, $r = \frac{d}{m}$, and $\phi(x) = x\,h(x)$.

2. $D_m \hookrightarrow G$ if and only if

$$\phi(x) = x\frac{F(x^m)}{G(x^m)}. = x\frac{x^{rm} + a_{r-1}x^{(r-1)m} + \cdots a_1 x^m + 1}{1 + b_{r-1}x^m + \cdots b_1 x^{(r-1)m} + x^{rm}}$$

where r is determined as follows: i) $m \mid (d - 1)$ and $r = \frac{d-1}{m}$
ii) $m \mid d$ and $r = \frac{d}{m}$.

3. $A_4 \hookrightarrow G$: Then $d \equiv 1 \mod 2$ and $\phi(x)$ can be written as $\phi(x) = x\frac{F(x^4)}{G(x^4)}$

4. $S_4 \hookrightarrow G$: Then $(d, 6) = 1$ and $\phi(x)$ can be written as where $\phi(x) = x\frac{F(x^4)}{G(x^4)}$

5. $A_5 \hookrightarrow G$: Then $d \equiv 1, 11, 19, 29 \mod 30$ and $\phi(x)$ can be written as where $\phi(x) = x\frac{F(x^5)}{G(x^5)}$

We summarize all cases in the following table.

Rational functions V

G	m	δ	$\phi(x)$	$\deg p(x)$	$\deg q(x)$
C_m	$m \mid d$	$\frac{2d}{m} - 1$	$x\,\frac{p(x^m)}{q(x^m)}$	$d-1$	$< d$
C_m	$m \mid d$	$\frac{2d}{m} - 1$	$x\,\frac{p(x^m)}{q(x^m)}$	$d-1$	$< d$
C_m	$m \mid d-1$	$\frac{2d-1}{m} - 1$	$x\,\frac{p(x^m)}{q(x^m)}$	$d-1$	$< d$
C_m	$m \mid d+1$	$\frac{2d+1}{m} - 1$	$x\,\frac{p(x^m)}{q(x^m)}$	d	d
D_m	$m \mid d-1$	$\frac{d-1}{m}$	$x\,\frac{p(x^m)}{q(x^m)}$	d	d
D_m	$m \mid d/2$	0	$x\,\frac{p(x^m)}{q(x^m)}$	$d-1$	d
D_m	$m \mid d+1$	$\frac{d+1}{m} - 1$	$x\,\frac{p(x^m)}{q(x^m)}$	d	d
A_4	$d \equiv 1 \bmod 2$	$\left\lfloor \frac{d}{6} \right\rfloor$	$x\,\frac{p(x^4)}{q(x^4)}$	d	d
S_4	$(d,6) = 1$	$\left\lfloor \frac{d}{12} \right\rfloor$	$\frac{p(x^4)}{q(x^4)}$	d	d
A_5	$d \equiv 1 \bmod 30$	$\left\lfloor \frac{d}{30} \right\rfloor$	$\frac{p(x^5)}{q(x^5)}$	d	d
A_5	$d \equiv 11 \bmod 30$	$\left\lfloor \frac{d}{30} \right\rfloor$	$\frac{p(x^5)}{q(x^5)}$	d	d
A_5	$d \equiv 19 \bmod 30$	$\left\lfloor \frac{d}{30} \right\rfloor$	$\frac{p(x^5)}{q(x^5)}$	d	d
A_5	$d \equiv 29 \bmod 30$	$\left\lfloor \frac{d}{30} \right\rfloor$	$\frac{p(x^5)}{q(x^5)}$	d	d

Invariants of rational functions

The action of $\mathrm{PGL}_2(k)$ on V_d extends naturally to an action on $\mathcal{F}_d$. For each $\sigma \in \mathrm{PGL}_2(k)$ such that $\begin{pmatrix} a & b \\ c & e \end{pmatrix}$ we have

$$\mathrm{PGL}_2(k) \times \mathcal{F}_d \to \mathcal{F}_d$$
$$\left(\begin{pmatrix} a & b \\ c & e \end{pmatrix}, f(x,y) \right) \to f^\sigma := \sigma^{-1} f \sigma,$$

Let $\phi := \frac{\mathbf{F}_0}{\mathbf{F}_1}$ and $\psi := \frac{\mathbf{G}_0}{\mathbf{G}_1}$ be degree $d \geq 2$ rational functions. By **??**, ϕ and ψ are k-equivalent if and only if there is $\sigma = \begin{pmatrix} a & b \\ c & e \end{pmatrix} \in \mathrm{PGL}_2(k)$ such that

$$G_0 = e F_0^\sigma - b F_1^\sigma \quad \text{and} \quad G_1 = -c F_0^\sigma + a F_1^\sigma. \tag{4}$$

For $d \geq 2$ and $\phi(x,y) = \frac{\mathbf{F}_0(x,y)}{\mathbf{F}_1(x,y)}$, define its **associated pair of binary forms**

$$\mathcal{I}_f := y\, \mathbf{F}_0 - x\, \mathbf{F}_1 \quad \text{and} \quad \mathcal{J}_f := \frac{\partial\, \mathbf{F}_0}{\partial x} + \frac{\partial\, \mathbf{F}_1}{\partial y} \tag{5}$$

Notice that $\mathcal{I}_f \in V_{d+1}$ and $\mathcal{J}_f \in V_{d-1}$.

Theorem
Let $f, g \in \mathcal{F}_d$ and $\sigma \in \mathrm{PGL}_2(k)$. Then $g = f^\sigma$ if and only if

$$\mathcal{I}_g = \mathcal{I}_f^\sigma \quad \text{and} \quad \mathcal{J}_g = \mathcal{J}_f^\sigma.$$

Since the pair of binary forms $(\mathcal{I}_f, \mathcal{J}_f)$ determines the rational function f, we want to use the classical theory of binary forms to determine invariants for f. Define the map

The action of $GL_2(k)$ on V_d induces an action of $GL_2(k)$ in $V_{d+1} \oplus V_{d-1}$. To determine the isomorphism classes of degree d rational functions we have to determine the ring of invariants of $V_{d+1} \oplus V_{d-1}$.

Let V be an SL_2-module and $\mathcal{O}(V)$ the algebra of polynomial functions on V. $SL_2(k)$ acts on $\mathcal{O}(V)$ via

$$M \cdot p(f_1, \ldots, f_r) \to p(M^{-1}f_1, \ldots, M^{-1}f_r),$$

for every $M \in SL_2(k)$. An invariant of V is an element $J \in \mathcal{O}(V)$ such that $MJ = J$, for all $M \in SL_2(k)$. The set of invariants is denoted by $\mathcal{O}(V)^{SL_2}$.

A transvectant $(J, I)_l$ is called **irrelevant** if there exist J_1, I_2, I_1, I_2 and I_1, I_2 such that

$$I = I_1 + I_2, \quad J = J_1 + I_2, \quad I = I_1 \cdot I_2,$$

and $I_1 \leq \mathrm{ord} J_1, \mathrm{ord} I_1,$ and $I_2 \leq \mathrm{ord} I_2, \mathrm{ord} I_2.$

Let V and W be two SL_2-modules whose covariants are finitely generated. Assume

$$J_1, \ldots, J_r : \qquad \text{are the generators of the covariants of } V$$
$$I_1, \ldots, I_s : \qquad \text{are the generators of the covariants of } W.$$

Theorem (Clebsch)

Then the covariants of $V \oplus W$ are also finitely generated. Moreover, a finite generating system of $V \oplus W$ can be chosen from the set of all transvectants

$$[J, I]_l, \quad \text{for} \quad l \geq 0,$$

where J is a monomial in the J_j's and I a monomial in the I_j's. In other words, by the non-irrelevant transvectants $[J, I]_l$.

We illustrate the above theorem with examples of degree $d = 3, 4$.

Example (Cubics)

Let $d = 3$, $F \in V_4$ and $G \in V_2$. Generators of V_4 are

$$J = (F, F)_2, \quad J_2 = (F, F)_4, \quad J_3 = ((F, F)_2, F)_4,$$

and the generator of V_2 is $I_2 = (G, G)_2$. Then the non-irrelevant transvections are I_2, J_2, J_3, and

$$(J, G^2)_4, \ (F, G^2)_4, \quad (G^3, (F, J)_1)_6,$$

which we denote them by J'_3, J_4, and J_6 respectively.

Rational Cubics I

As an application of **??** we illustrate with rational functions of degree 3. In this case, the invariants of degree 3 rational functions are invariants of $V = V_4 \oplus V_2$. As an easy consequence of **??** for $d = 3$ we can determine such invariants in terms of transvections. We describe such invariants explicitly below.

Let $F(x,y) \in V_4$ and $G(x,y) \in V_2$ say

$$F = a_0 x^4 + a_1 x^3 y + a_2 x^2 y^2 + a_3 x y^3 + a_4 y^4$$
$$G = b_0 x^2 + b_1 x y + b_2 y^2 \tag{7}$$

It is well known that the ring of invariants $\mathcal{O}(V)^{SL_2}$ is generated by the invariants

$$I_2 = (G,G)_2, \qquad J_2 = (F,F)_4, \quad J_4 = ((F,F)_2, G^2)_4$$
$$I_3 = ((F,F)_2, F)_4, \quad J_3 = (F, G^2)_4, \quad J_6 = (G^3, (F, (F,F)_2)_1)_6$$

this follows from a result of Clebsch. The **nullcone** $\mathcal{N}_{(4,2)}$ is the common zero set of all homogeneous elements of positive degree in $\mathcal{R}_{(4,2)}$

Proposition
The ring of invariants $\mathcal{R}_{4,2}$ is generated by $I_2, J_2, I_3, J_3, J_4, J_6$ which satisfy the suzuki

$$J_6^2 = \frac{1}{108} I_2^3 J_2^3 - 18 I_2^3 I_3^2 - \frac{1}{24} I_2 J_2^2 J_3^2 - \frac{1}{6} I_3 J_3^3 + \frac{1}{2} I_2 I_3 J_3 J_4 + \frac{1}{4} J_2 J_3^2 J_4 - \frac{1}{4} I_2 J_2 J_4^2 - \frac{1}{2} J_4^3 \tag{8}$$

Rational Cubics II

Next we want to be able to directly compute the above invariants from a degree three rational function. Let $f(x,y) = \frac{F_0(x,y)}{F_1(x,y)}$, where

$$F_0(x,y) = A_0 x^3 + A_1 x^2 y + A_2 xy^2 + A_3 y^3,$$
$$F_1(x,y) = B_0 x^3 + B_1 x^2 y + B_2 xy^2 + B_3 y^3 \tag{9}$$

Then $\psi(f) = (\mathcal{I}_f, \mathcal{J}_f)$, where

$$\mathcal{I}_f = -B_0 x^4 + (A_0 - B_1) x^3 y + (A_1 - B_2) x^2 y^2 + (A_2 - B_3) xy^3 + A_3 y^4$$
$$\mathcal{J}_f = (3A_0 + B_1) x^2 + 2(A_1 + B_2) xy + (A_2 + 3B_3) y^2$$

Evaluated to $(\mathcal{I}_f, \mathcal{J}_f)$ we have the above invariants easily computed as polynomials in $k[A_0, \ldots, A_3, B_0, \ldots, B_3]$. Thus, the tuple of invariants of ϕ is defined as

$$\xi(\phi) = (I_2(\phi), J_2(\phi), I_3(\phi), J_3(\phi), J_4(\phi), J_6(\phi))$$

Explicit formulas are displayed in **??**.
Let us now compute the invariants of ϕ^M as defined in **??** and check if they are really SL$_2$-invariants. This action of GL$_2(k)$ on $\mathcal{F}_3$ induces an action on $k[A_0, \ldots, A_3, B_0, \ldots, B_3]$ which takes

$$(A_0, \ldots, A_3, B_0, \ldots, B_3) \to (A_0', \ldots, A_3', B_0', \ldots, B_3')$$

Rational Cubics III

$$A_0' = a^3eA_0 - a^3bB_0 + a^2ceA_1 - a^2bcB_1 + ac^2eA_2 - abc^2B_2 + c^3eA_3 - bc^3B_3$$

$$A_1' = 3a^2beA_0 - 3a^2b^2B_0 + ae(ae + 2bc)A_1 - ab(ae - 2bc)B_1 + ce(2ae + bc)A_2$$
$$\quad - bc(2ae + bc)B_2 + 3c^2e^2A_3 - 3bc^2eB_3$$

$$A_2' = 3ab^2eA_0 - 3ab^3B_0 + be(2ae + bc)A_1 - b^2(2ae + bc)B_1 + e^2(ae + 2bc)A_2$$
$$\quad - be(ae + 2bc)B_2 + 3ce^3A_3 - 3bce^2B_3$$

$$A_3' = b^3eA_0 - b^4B_0 + b^2e^2A_1 - b^3eB_1 + be^3A_2 - b^2e^2B_2 + e^4A_3 - be^3B_3$$

$$B_0' = -a^3cA_0 + a^4B_0 - a^2c^2A_1 + a^3cB_1 - ac^3A_2 + a^2c^2B_2 - c^4A_3 + ac^3B_3$$

$$B_1' = -3a^2bcA_0 + 3a^3bB_0 - ac(ae + 2bc)A_1 + a^2(ae + 2bc)B_1 - c^2(2ae + bc)A_2$$
$$\quad + ac(2ae + bc)B_2 - 3c^3eA_3 + 3ac^2eB_3$$

$$B_2' = -3ab^2cA_0 + 3a^2b^2B_0 - bc(2ae + bc)A_1 + ab(2ae + bc)B_1 - ce(ae + 2bc)A_2$$
$$\quad + ae(ae + 2bc)B_2 - 3c^2e^2A_3 + 3ace^2B_3$$

$$B_3' = -b^3cA_0 + ab^3B_0 - b^2ceA_1 + ab^2eB_1 - bce^2A_2 + abe^2B_2 - ce^3A_3 + ae^3B_3$$

Thus, we computationally have proved the following:

Proposition

Invariants of ϕ^M as polynomials in $k[A_0', \ldots, B_3']$ are given by

$$
\begin{aligned}
I_2(\phi^M) &= \lambda^2 \, I_2(\phi), & J_2(\phi^M) &= \lambda^2 \, J_2(\phi), & I_3(\phi^M) &= \lambda^3 \, I_3(\phi) \\
J_3(\phi^M) &= \lambda^3 \, J_3(\phi), & J_4(\phi^M) &= \lambda^4 \, J_4(\phi), & J_6(\phi^M) &= \lambda^6 \, J_6(\phi)
\end{aligned}
\tag{10}
$$

where $\lambda = \det(M)^2$. Two cubic rational functions $f, g \in \mathcal{F}_3$ are equivalent if and only if there exists some $c \in k^*$ such that

$$
(I_2(g), J_2(g), I_3(g), J_3(g), J_4(g), J_6(g)) = \left(c^2 I_2(f), c^2 J_2(f), c^3 I_3(f), c^3 J_3(f), c^4 J_4(f), c^6 J_6(f) \right)
$$

Hence, the isomorphism classes of degree three rational functions correspond to points in the weighted projective space $\mathbb{WP}_\omega^5$, for weights $\omega = (2, 2, 3, 3, 4, 6)$.

Rational Cubics I

Let $f \in \mathrm{Rat}_3^1$, $G = \mathrm{Aut}(f)$, and W its set of fixed points. Then we have the following result:

If $C_2 \hookrightarrow G$ then one of the following cases occur:

1. If C_2 fixes two points in W then $f(x,y) = \frac{y^3 + ayx^2}{by^2x + x^3}$, for $a, b \in k$, $ab \neq 1$, and $\mathcal{L}(C_2, 1)$ is a dimension 2 irreducible algebraic variety birationaly parametrized by $(u, v) \mapsto (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, 0)$, where $u = (a+b)^2$, $v = a - b$. If $a \neq -b$ then $u = -\frac{\xi_0}{2}$, $v = -3\frac{\xi_3}{\xi_0}$, and

$$\xi_1 = \frac{3\xi_3^2 - 4\xi_0^2}{2\xi_0^2}, \quad \xi_2 = \frac{3}{4}\xi_3\frac{4\xi_0^2 + \xi_3^2}{\xi_0^3}, \quad \xi_4 = \frac{4\xi_0^2 + 3\xi_3^2}{6\xi_0},$$

   otherwise, $\xi_0 = \xi_3 = \xi_4 = \xi_5 = 0$, and $6\xi_2^2 - (2 + \xi_1)(8 + \xi_1)^2 = 0$.

2. If C_2 fixes one point in W then $f(x,y) = \frac{ay^2x + x^3}{y^3 + byx^2}$, for $a, b \in k$, $ab \neq 1$ and $\mathcal{L}(C_2, 2)$ is a dimension 2 irreducible algebraic variety birationaly parametrized by $(u, v) \mapsto (\xi_0, \xi_1, 0, 0, \xi_4, \xi_5)$, where $u = a + b$, $v = ab$ $u = \frac{\xi_0 - 4\xi_1 - 16}{8}$, $v = \frac{\xi_0 + 12\xi_1 - 24}{8}$, and

$$\xi_4 = -\frac{1}{96}\left(3\xi_0^2 + 8\left(\xi_1 - 24\right)\xi_0 + 48(\xi_1 - 8)^2\right), \quad \xi_5 = -\frac{1}{256}(\xi_0 + 4\xi_1 - 32)\left(\xi_0^2 - 8(\xi_1 + 8)\xi_0 + 16(\xi_1 - 8)^2\right).$$

Rational Cubics II

If $V_4 \hookrightarrow G$ then we can take $f(x,y) = \frac{ay^2x \pm x^3}{y^3 \pm ayx^2}$, for $a^2 \neq 1$. In particular, $\xi(f) = \left[\xi_0 : \xi_1 : 0 : 0 : -\frac{1}{3}\xi_0\xi_1 : 0\right]$
such that $6\xi_2^2 - (2 + \xi_1)(8 + \xi_1)^2 = 0$. The locus $\mathcal{L}_4$ of such functions in Rat_3^1 is an irreducible algebraic curve birationaly parametrized via

$$a \mapsto \left(\xi_0, \xi_1, 0, 0, -\frac{1}{3}\xi_0\xi_1, 0\right)$$

where $\xi_0 = 2(a+3)^2$ and $\xi_1 = 2(a-1)^2$.

If $C_3 \hookrightarrow G$ then we can take $f(x,y) = \frac{y^3 + ax^3}{ay^2x}$, for $a \in k^\star$. In particular,

$$\xi(f) = \left[-12\xi_1 : \xi_1 : \xi_2 : -24\xi_2 : -2\xi_1(2\xi_1 + 243) - 216\xi_2 : 27\xi_1(81\xi_1 + 32\xi_2)\right]$$

such that $0 = 8\xi_1^3 - 3\left(9\xi_1 + 4\xi_2\right)^2$. The locus $\mathcal{L}_3$ of such functions in Rat_3^1 is an irreducible curve birationaly parametrized via

$$a \mapsto \left(-12\xi_1, \xi_1, \xi_2, -24\xi_2, -2\xi_1(243 + 2\xi_1) - 216\xi_2, -27\xi_1(81\xi_1 + 32\xi_2)\right)$$

where $\xi_1 = \frac{1}{6}a^2$, $\xi_2 = -\frac{1}{72}a^2(2a + 27)$.

If $A_4 \hookrightarrow G$ then $f(x,y) = \frac{y^3 - 3x^3}{-3y^2x}$ and. $\xi(f) = \left[-18 : \frac{3}{2} : -\frac{33}{8} : 99 : 153 : \frac{1701}{4}\right]$.

If $C_4 \hookrightarrow G$ or $D_4 \hookrightarrow G$ then $f(x,y) = \frac{x^3}{y^3}$ and $\xi(f) = \left[18 : \frac{1}{2} : 0 : 0 : -3 : 0\right]$

Thank you!!