Orbits of Theta Characteristics

Computation and Theory
Based on arXiv:2404.09890 and thesis
Joint with Harry Braden

Linden Disney-Hogg

University of Edinburgh / University of Leeds

Outline

1. Motivation and Background
2. Group Cohomology
3. Numerical Computations
4. Hurwitz Curves
5. Outlook

Motivation and Background

Background - Theta Characteristics

Fix a smooth compact connected Riemann surface $\mathcal{C}$.

A theta characteristic on $\mathcal{C}$ is a line bundle $L \to \mathcal{C}$ such that $L^2 := L \otimes L$ is isomorphic to the canonical bundle $K_{\mathcal{C}}$. The set of characteristics $S(\mathcal{C})$ is an affine space over $\mathbb{Z}_2$ modelled on $\mathrm{Pic}^0(\mathcal{C})[2] \cong H_1(\mathcal{C}, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$.

Part distinguished by their parity; a theta characteristic L is called odd/even if $\dim H^0(\mathcal{C}, L)$ is odd/even. There are exactly $2^{g-1}(2^g - 1)$ odd theta characteristics and $2^{g-1}(2^g + 1)$ even theta characteristics.

Example: $\mathcal{C} = \mathbb{P}^1$
$K_{\mathbb{P}^1} = \mathcal{O}(-2)$, and $S(\mathcal{C}) = \{\mathcal{O}(-1)\}$. $\dim H^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, so characteristic is even.

Example: Tangent Hyperplanes
When $\mathcal{C}$ nonhyperelliptic, tangent planes to canonical embedding give effective theta characteristics.

Background - Action of Automorphisms

Assume $g(\mathcal{C}) \geqslant 2$. Fix a basis of $H_1(\mathcal{C}, \mathbb{Z})$ with intersection matrix $J = \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right)$ and denote the rational representation with $\rho_r : \mathrm{Aut}(\mathcal{C}) \to \mathrm{Sp}_{2g}(\mathbb{Z})$. $\mathrm{Aut}(\mathcal{C})$ permutes theta characteristics, preserving parity.

Theorem (Kallel, Sjerve, 2010)
There is an isomorphism $S(\mathcal{C}) \cong \mathbb{Z}_2^{2g}$ such that the parity of $x = (u, v) \in \mathbb{Z}_2^{2g}$ is $q(x) = u \cdot v$ and the action of $f \in \mathrm{Aut}(\mathcal{C})$ is

$$f : x \mapsto \bar{R}^T x + v \mod 2,$$

where $R = \rho_r(f)$, $\bar{R}$ is its mod-2 reduction, and $v_i = \sum_{j<j'} R_{ji} R_{j'i} J_{jj'}$.

Changing isomorphism by translating by $y \in \mathbb{Z}_2^{2g}$, i.e. denoting $x = x' + y$, the transform is then $x' \mapsto R^T x' + v_y$ where

$$v_y = v + (\bar{R}^T - I)y.$$

Background - Invariant Characteristics: What Was Known

Theorem (Atiyah, 1971)
Any affine action over $\mathbb{Z}_2$ preserving a quadratic form with nondegenerate associated bilinear has a fixed point. Hence, every automorphism leaves at least one characteristic invariant.

Theorem ([KS10])
The number of characteristics fixed by f is $2^{\dim \ker(\bar{R}^T - I)}$. In particular, when f has odd order, it fixes a unique characteristics iff $g(\mathcal{C}/\langle f \rangle) = 0$.

Theorem (Biswas, Gadgil, Sankaran, 2007, [KS10])
The hyperelliptic involution is the only automorphism to fix all characteristics.

Example: (Burns, 1981, Dolgachev 1997, [KS10])
Klein's curve, the Hurwitz curve (i.e. $|\mathrm{Aut}(\mathcal{C})| = 84(g-1)$) in genus 3 with $\mathrm{Aut}(\mathcal{C}) = \mathrm{PSL}_2(7)$, has a unique invariant characteristic.

Example: (Braden, Northover, 2012, Braden, DH, 2022)
Bring's curve, the curve in genus 4 with $\mathrm{Aut}(\mathcal{C}) = S_5$ of maximal order, has a unique invariant characteristic.

Motivation - Questions and Applications

Questions:

- Can we determine necessary and sufficient conditions for a Unique Invariant Characteristic (UIC) under action of an even-order automorphism? Under any $G \leqslant \mathrm{Aut}(\mathcal{C})$?
- Are there infinitely many curves with a UIC?
- Answers for exceptionally symmetric curves?
- How to uncover what the right theorems should be?

Applications:

- Theta characteristics equivalent to spin structures on a Riemann surface [Ati71], relevant for TQFT (Masbaum, 1997) and Floer homology (Lin, 2022).
- Find effective even theta characteristics (Beauville, 2013), and so possible applications to Schottky problem.
- Constructing Fano threefolds (Chelstov, Li, Ma'u, Pinardin, 2024).

Group Cohomology

Affine Representation as Group Cohomlogy

An affine representation of G on V which acts multiplicatively via $\rho : G \to \mathrm{GL}(V)$ is

$$G \times V \to V,$$
$$(g, x) \mapsto g \cdot x := \rho(g)x + v(g).$$

Lemma (Beckett, ?)
v determines an element of the group cohomology $H^1(G, V)$, zero iff the affine rep is equivalent under a translation of V to the linear action.

Take $\rho = \overline{\rho_r}^T$ and $V = H_1(\mathcal{C}, \mathbb{Z}_2)$, the action on characteristics gives a cocycle, trivial in $H^1(G, V)$ iff there exists a characteristic invariant under G, and if invariant characteristics exist, there are $2^{\dim H^0(G,V)}$ many.

Cohomology of Cyclic Groups

Example: Cyclic Groups
$G = \langle f \rangle$, $A := \rho(f) = \bar{R}^T$, $o(f) = n > 1$.

$$H^0(G, V) = \mathrm{Ker}(A - I), \quad H^1(G, V) = \mathrm{Ker}(I + \cdots + A^{n-1}) / \mathrm{Im}(A - I).$$

When n odd, $(A - I)$ is invertible iff $\sum_{k=0}^{n} A^{k-1} = 0$ iff $g(\mathcal{C}/\langle f \rangle) = 0$ (Ries, 1993), recreating [KS10].

When n is even, again using Ries, necessary and sufficient condition is $\forall l$ such that $2^l | n$, $g\left(\mathcal{C}/\left\langle f^{(2^l)} \right\rangle\right) = 0$ (Braden, DH, 2024).

Subnormal Groups

Given $N \lhd G$ have the inflation-restriction exact sequence

$$0 \to H^1(G/N, V^N) \to H^1(G, V) \to H^1(N, V)^{G/N}.$$

Proposition (Braden, DH, 2024)
If there exists $f \in G$ such that f has odd order, $\langle f \rangle$ is subnormal in G, and $g(\mathcal{C}/\langle f \rangle) = 0$, then $\mathcal{C}$ has a unique theta characteristic invariant under the action of G. We will call a curve with this property Subnormal Odd Cyclic (SOC).

The captures any even cyclic with a UIC.

Corollary (Braden, DH, 2024)
There are infinitely many curves, both hyperelliptic and non-hyperelliptic, with a unique invariant characteristic.

Proof.
Consider $y^2 = x^{2g+1} - 1$ with $\mathrm{Aut} = C_{2g+1} \times C_2$ and $x^m y^n + y^m + x^n = 0$ (coprime m, n where $p := m^2 - mn + n^2 > 7$ is prime $\equiv 1 \bmod 3$) with $\mathrm{Aut} = C_p \rtimes C_3$. □

Numerical Computations

Tables of Orbits

f	$\overline{\mathrm{Aut}}$, $\boldsymbol{c}$	Odd	Even	I
$y^2-(x^2-1)(x^2-a)(x^2-b)$	$C_2,\ (1;2^2)$	2_3	$1_4, 2_3$	4
$y^2-(x^2-1)(x^2-a)(x^2-a^{-1})$	$V_4,\ (0;2^5)$	$2_1, 4_1$	$1_2, 2_2, 4_1$	2
$y^2-(x^5-1)$	$C_5,\ (0;5^3)$	$1_1, 5_1$	5_2	1
$y^2-(x^6-ax^3+1)$	$S_3,\ (0;2^2,3^2)$	6_1	$1_1, 3_3$	1
$y^2-(x^6-1)$	$D_6,\ (0;2^3,3)$	6_1	$1_1, 3_1, 6_1$	1
$y^2-x(x^4-1)$	S_4	6_1	$4_1, 6_1$	0

126 plane forms from [Bol87, Bar12, Wim95a, Sha07, MP21, Wim95b, KR89, Swi16, BB16, MSSV02, Lef21, Hid17] in genera $g \leqslant 9$. Sage gives ρ_r (Bruin, Sijsling, Zotine, 2019), signatures use LMFDB. 52 with a UIC, 5 exceptions to SOC.

Machine Learning

Code from Behn, Rodriguez, Rojas, 2013 gets ρ_r from the data of $(G, \boldsymbol{c})$ and choice of generating vector, provided $g(\mathcal{C}/G) = 0$. Lots of such data known (1326 up to topological equivalence in genera $g \leqslant 11$).

Machine classification[1] using features genus, group order, whether the group action was large (in the sense $|G| > 4(g-1)$), the maximum power of 2 dividing the group order, the number of involutions in the group, the number of involutions up to conjugacy, the number of odd and even entries in the signature, the maximum entry of the signature, and the dimension of the corresponding family.

Achieved accuracy $\approx$ 93% in cross-validation when predicting if group action gave UIC, suggesting signature-level data should give reasonable estimate of existence of UIC. Estimating feature importance guides which aspects to ignore when formulating conjectures.

[1] Thanks to Jacob Bradley

Hurwitz Curves

Predictions

Table 2: Prediction, $I = 1$, all simple Hurwitz groups order $< 10^6$

G	g	$I = 1$
$PSL_2(7)$	3	True
$PSL_2(8)$	7	False
$PSL_2(13)$	14	True
$PSL_2(27)$	118	True
$PSL_2(29)$	146	True
$PSL_2(41)$	411	False
$PSL_2(43)$	474	True
J_1	2091	False
$PSL_2(71)$	2131	False
$PSL_2(83)$	3404	True
$PSL_2(97)$	5433	False
J_2	7201	False
$PSL_2(113)$	8589	False
$PSL_2(125)$	11626	True

Dolgachev's Exact Sequence

Proposition (Dolgachev, 1997)
Write $\mathrm{Pic}(G;\mathcal{C}) = \mathrm{Div}(\mathcal{C})^G/[\mathcal{M}(\mathcal{C})^\times]^G$, *have*

$$0 \to \mathrm{Hom}(G,\mathbb{C}^\times) \to \mathrm{Pic}(G;\mathcal{C}) \to \mathrm{Pic}(\mathcal{C})^G \to H^2(G,\mathbb{C}^\times) \to 0.$$

Proposition (Dolgachev, 1997)
Suppose G has signature $(0; c_1,\ldots,c_r)$,*take* $d_0 = 1$, $d_1 = \gcd(c_i)$, $d_2 = \gcd(c_i c_j)$ *etc., and* $N = \mathrm{lcm}(c_i)$. *Then*

$$\mathrm{Pic}(G;\mathcal{C}) \cong \mathbb{Z} \oplus \left[\bigoplus_{i=1}^{r-1} \mathbb{Z}/(d_i/d_{i-1})\mathbb{Z}\right], \quad K_{\mathcal{C}} = N\left(r - 2 - \sum_{i=1}^{r} \frac{1}{c_i}\right)\gamma,$$

where γ *generates the* $\mathbb{Z}$ *factor.*

Example - Hurwitz Curves with G Simple
$\mathrm{Hom}(G,\mathbb{C}^\times) = 0$, $H^2(G,\mathbb{C}^\times)$ known cyclic group,
$\boldsymbol{c} = (0; 2, 3, 7) \Rightarrow \mathrm{Pic}(G;\mathcal{C}) = \mathbb{Z}K_{\mathcal{C}}$.

Hurwitz Curves with Simple Automorphism Group

Proposition (Braden, DH, 2024)
Given a Hurwitz curve $\mathcal{C}$ with simple automorphism group G, and $g \in G$ generating the stabiliser group $G_{P_1} \cong C_2$, either:

1. $H^2(G, \mathbb{C}^\times)$ *is cyclic of odd order, and* $\mathcal{C}$ *has no invariant characteristics,*
2. $H^2(G, \mathbb{C}^\times)$ *is cyclic of even order, the lifting order of* g *in* $2 \cdot G$ *is 2, and* $\mathcal{C}$ *has no invariant characteristics, or*
3. $H^2(G, \mathbb{C}^\times)$ *is cyclic of even order, the lifting order of* g *in* $2 \cdot G$ *is 4, and* $\mathcal{C}$ *has exactly one invariant characteristic.*

Proposition (Braden, DH, 2024)
If $G = \mathrm{PSL}_2(q)$ *(*q *odd),* J_2*, or* Ru*,* $\mathcal{C}$ *has a UIC.*

If $G = A_n$ the answer is unknown but have criterion to determine, otherwise $\mathcal{C}$ has no invariant characteristics.

Outlook

Outlook

- Necessary condition for a unique invariant characteristics?
- What lies beyond SOC?
- Structure of the orbit decomposition.
- Connecting the Dolgachev exact sequence to the group cohomology approach.
- UICs for A_n Huwritz curves?
- Extension to rth-roots of the canoncial bundle.
- Machine scored 64% on simple Hurwitz curves - danger of extrapolation.

Extras - A_n Hurwitz curves

Let $\gamma_{i=1,2,3}$, be generators order $2, 3, 7$ in corresponding generating vector, denote $f_i = |\text{Fix}(\gamma_i)|$. $\exists h \in \mathbb{Z}_{\geqslant 0}$ such that

$$n = 84(h-1) + 21f_1 + 28f_2 + 36f_3.$$

Lemma (Braden, DH, 2024)

$$UIC \Leftrightarrow h - 1 + f_1 + f_2 + f_3 \not\equiv 0 \mod 2.$$

First 5 entries from Etayo Gordejuela, Martinez, 2005.

$(n; h, f_1, f_2, f_3)$	$I = h - 1 + f_1 + f_2 + f_3 \mod 2$
$(15; 0, 3, 0, 1)$	1
$(21; 0, 1, 3, 0)$	1
$(22; 0, 2, 1, 1)$	1
$(28; 0, 4, 1, 0)$	0
$(29; 0, 1, 2, 1)$	1

References i

- M. F. Atiyah, *Riemann surfaces and spin structures*, Annales scientifiques de l'École Normale Supérieure, Serie 4 **4** (1971), no. 1, 47–62.
- F. Bars, *On the automorphism group of genus 3 curves*, Surveys in Mathematics and Mathematical Sciences **2** (2012), no. 2, 83–124.
- E. Badr and F. Bars, *Automorphism groups of nonsingular plane curves of degree 5*, Communications in Algebra **44** (2016), no. 10, 4327–4340.
- O. Bolza, *On binary sextics with linear transformations into themselves*, American Journal of Mathematics **10** (1887), no. 1, 47–70.
- R. A. Hidalgo, *About the Fricke-Macbeath curve*, arXiv:1703.01869, 2017.

References ii

- E. Kani and M. Rosen, *Idempotent relations and factors of Jacobians*, Mathematische Annalen **284** (1989), no. 2, 307–327.
- S. Kallel and D. Sjerve, *Invariant spin structures on Riemann surfaces*, Annales de la Faculté des Sciences de Toulouse **19** (2010), 457–477.
- S. Lefschetz, *On certain numerical invariants of algebraic varieties with application to abelian varieties*, Transactions of the American Mathematical Society **22** (1921), no. 4, 407–482.
- N. Müller and R. Pink, *Hyperelliptic curves with many automorphisms*, International Journal of Number Theory **18** (2021), no. 4, 913–930.

References iii

K. Magaard, T. Shaska, S. Shpectorov, and H. Voelklein, *The locus of curves with prescribed automorphism group*, Communications on Arithmetic Fundamential Groups and Galois Theory, Kyoto Technical Report Series, 2002.

T. Shaska, *Some open problems in computational algebraic geometry*, Albanian Journal of Mathematics **1** (2007), no. 4, 297–319.

D. Swinarski, *Equations of Riemann surfaces with automorphisms*, arXiv:1607.04778, 2016.

A. Wiman, *Über die hyperelliptischen curven und diejenigen vom geschlechte p= 3, welche eindeutigen transformationen in such zulassen*, Bihang till Kongl. Svenska vetenskaps-akademiens handlingar **21** (1895), 1–23.

——— , *Ueber die algebraischen curven von den geschlechten p= 4, 5 und 6, welche eindeutigen transformationen in sich besitzen*, Bihang till Kongl. Svenska vetenskaps-akademiens handlingar **21** (1895), 1–41, English translation available at https://arxiv.org/abs/2204.01656.