Families of Riemann Surfaces with Large Groups of Automorphisms

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AMS Special Session on Automorphisms of Riemann Surfaces and Related Topics

The Grounds

Given an orientable, **closed** surface X of genus $\mathbf{g \geq 2}$ The equivalence:

- $(X, \mathcal{M}(X)$, complex atlas)
  $\mathcal{M}(X) = \langle x, y \rangle$, $p(x, y) = 0$, the field of meromorphic functions on X
- $X \cong \frac{\mathbb{H}}{\Delta}$, with Δ a (cocompact, torsion free) **Fuchsian group**
  Γ discrete subgroup of $PSL(2, \mathbb{R})$
  Surface Fundamental Group $\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \Pi[a_i, b_i] = 1 \rangle$. Riemann Uniformization Theorem (Koebe)
- $(X, \mathcal{M}(X)$, **complex curve**) ($\mathcal{M}(X) = \mathbb{C}[x, y]/p(x, y)$, the field of rational functions on X)
  The curve X given by the polynomial $p(x, y)$ and the meromorphic function $x : X \to \widehat{\mathbb{C}}$.

Riemann Ph.D. Thesis: *Foundations for a general theory of functions of a complex variable*, 1851

Hurwitz, *Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. 103, (1891), 1-60,

Poincaré Papers in Acta Mathematica on Fuchsian Groups, 1895 -

Koebe: Riemann Uniformization Theorem

frame We construct a **Riemann Surface** for the function $R(x, y)$, with y is a locally defined function such that $F(x, y) = r_0(x)y^n + r_i(x)y^{n-1} + \cdots + r_n(x) = 0$, $r_i(x)$ rational:

- Take n (n the degree of $F(x, y)$) copies of the complex plane with 'branch cuts'
- Each copy of the cut plane defines a branch of y
- Glue, using analytical continuation, the cut planes along the branch cuts (keeping y continuous)
- Compactify by adding points at infinity and points corresponding to branch points
- Resolve singularities

We have a Riemann surface S (with certain **conformal structure**) with:

- Two important meromorphic functions $x : S \to \widehat{\mathbb{C}}$ and $y : S \to \widehat{\mathbb{C}}$
- The branch cuts decompose the surface into n polygons. The function x lifts the base cut plane bijectively to the polygons.
- x, y generate the field of meromorphic functions on X, $\mathbb{C}(S)$, is a degree n extension of the rational functions field $\mathbb{C}(S) = \mathbb{C}(x, y)/\langle F(x, y)\rangle$

S is a surface whose topological genus g is by the Riemann-Hurwitz formula
$2 - 2g = n(2 + \sum \frac{1-q_i}{n})$,
with q_i the multiplicities of the branch points.

(Riemann Ph.D. Thesis: *Foundations for a general theory of functions of a complex variable*, 1851, Klein's construction of his Quartic 1879, Hurwitz 1891

Number Fields, Triangular Groups and Dessins d'Enfants

Belyi Th: A plane complex curve X is defined over a number field iff there is a finite N-sheeted orbifold-covering = meromorphic function $\beta : X \to \widehat{\mathbb{C}}$ of the Riemann sphere **ramified on at most three points** $\{0, 1, \infty\}$ (the meromorphic function β is Belyi function).
In the case of Klein's Quartic, a 7-sheeted orbifold-covering of the Riemann sphere ramified at three points ($y^7 = x^2(x-1)$).

The meromorphic function β induces a cell-decomposition $\mathcal{H}$ of the Riemann surface X: the dessin d'enfant (map or hypermap). The preimages of 0 providing the *hypervertices,* the preimages of 1 the *hyeredges* and the preimages of ∞ the *hyperfaces*.
In the case of Klein's Quartic, the tessellation is the well-known tessellation with 168 triangles, each one representing an element in $PSL(2, 7)$.

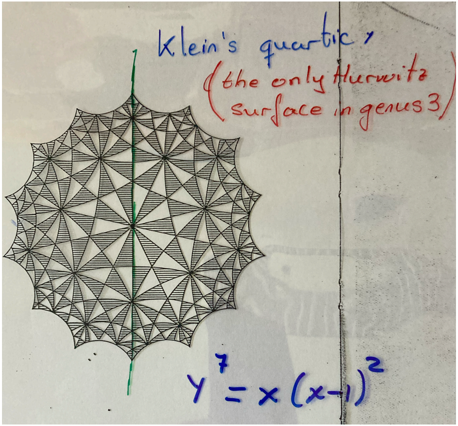
Translating into Fuchsian groups: $\beta : \mathbb{H}/\Gamma_g \to \mathbb{H}/\Delta(l, m, n)$, where $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.
The dessin has type (l, m, n), and (orbifold-covering) monodromy

$$\theta_\beta : \Delta(l, m, n) \to G = Mon(\mathcal{H})$$

In the case of Klein's Quartic, $\theta_y : \Delta(2, 3, 7) \to PSL(2, 7) = Mon(\mathcal{H})$.

Only interested in uniform dessin: $\Gamma_g = H = \theta_\beta^{-1}(Stb(1))$ (a surface group).

Jones-SIngerman Proc London Math Soc, 1978 (Singerman 1976)

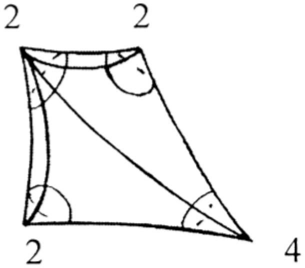


Klein *"Uber die Transformationen siebenter Ordnung der elliptischen Funktionen*
Math. Ann. 14, 428-471, 1879.

Fuchsian Groups

- Δ (cocompact) discrete subgroup of $PSL(2, \mathbb{R})$
- A (compact) Riemann Surface (Orbifold) of genus $g \geq 2$ $\qquad X = \frac{\mathbb{H}}{\Delta}$
- Δ has presentation:
  generators: $x_1, ..., x_r, a_1, b_1, ..., a_h, b_h$
  relations: $x_i^{m_i}, i = 1 : r,\ x_1 ... x_r a_1 b_1 a_1^{-1} b_1^{-1} ... a_h b_h a_h^{-1} b_h^{-1}$
  x_i: generator of the maximal cyclic subgroups of Δ
- $X = \frac{\mathbb{H}}{\Delta}$: orbifold with r cone points and underlying surface of genus g
- Algebraic structure of Δ and geometric structure of X are determined by the signature $\qquad s(\Delta) = (h; m_1, \ldots, m_r)$
- Δ is the orbifold-fundamental group of X.

- Area of Δ: area of a fundamental region P
  $\mu(\Delta) = 2\pi(2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$
  But for us $\mu(\Delta) = (2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$, (-Euler characteristic of the hyperbolic orbifold)
- **X hyperbolic equivalent to $P/\langle$pairing$\rangle$**
- **Poincaré's Th: $\Delta = \langle$pairing$\rangle$** (Maskit, 1971)
- **Riemann-Hurwitz Formula:** If Λ is a subgroup of finite index, N, of a Fuchsian group Δ, then $N = \frac{\mu(\Lambda)}{\mu(\Delta)}$ (Euler characteristic is multiplicative under coverings)
- RUT: Any Riemann surface of genus $g \geq 2$ is uniformized by a surface Fuchsian group
  $\Gamma_g = \langle a_1, b_1, ..., a_g, b_g ; a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} \rangle$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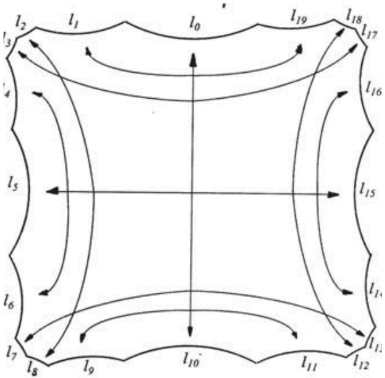
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Automorphisms and Morphisms of RS

G finite group of automorphisms of $X_g = \mathbb{H}/\Gamma$, Γ a surface group iif there exist Δ Fuchsian/NEC group and epimorphism $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma$

θ is the monodromy of the (regular) covering $f : \mathbb{H}/\Gamma \to \mathbb{H}/\Delta$

$$\begin{array}{ccc} & & \mathbb{H} \\ & \swarrow & \\ X = \mathbb{H}/\Gamma & & \downarrow \\ & \searrow & \\ & & X/G = \mathbb{H}/\Delta \end{array}$$

Δ: lifting to $\mathbb{H}$ of G

An automorphism of X will fix the class of the uniformizing Fuchsian/NEC group

A morphism $f : X = \mathbb{H}/\Lambda \to Y = \mathbb{H}/\Delta$, given by the group inclusion $i : \Lambda \to \Delta$
Covering f determined by monodromy $\theta : \Delta \to \Sigma_{|\Delta:\Lambda}, \Lambda| = \theta^{-1}(STb(1))$
(symbol $\leftrightarrow$ Λ-coset $\leftrightarrow$ sheet for f)

Theorem (Singerman 1971) Λ (and so i) determined θ (and Δ): If $s(\Delta) = (h; m_1, \ldots, m_r)$, then $s(\Lambda) = (h'; m'_{11}, \ldots, m'_{1s_1}, \ldots, m'_{r1}, \ldots, m'_{rs_r})$ iff $\theta : \Delta \to \Sigma_{|\Delta:\Lambda|}$ s.t.

i) Riemann-Hurwitz $\frac{\mu(\Lambda)}{\mu(\Delta)} = |\Delta : \Lambda|$

ii) $\theta(x_i)$ product of s_i cycles each of length $\frac{m_i}{m'_{i1}}, \ldots, \frac{m_i}{m'_{is_i}}$

Analogous result for NEC group & Klein surfaces Singerman 1974, Hoare 1990, Pride 1990

Let S be a compact Riemann surface of genus $g \geqslant 2$. We denote by $H^1(S,\mathbb{C})^*$ the dual of the g-dimensional complex vector space of 1-forms on S, and by $H_1(S,\mathbb{Z})$ the first integral homology group of S.
The Jacobian variety of S, defined by

$$JS = H^1(S,\mathbb{C})^*/H_1(S,\mathbb{Z}),$$

is an irreducible principally polarized abelian variety of dimension g. The relevance of the Jacobian variety lies, partially, in Torelli's theorem, which establishes that two Riemann surfaces are isomorphic if and only if the corresponding Jacobian varieties are isomorphic as principally polarized abelian varieties.

If $H \leqslant \mathrm{Aut}(S)$ then the associated regular covering map $\pi : S \to S/H$ induces a homomorphism

$$\pi^* : JS_H \to JS$$

between the associated Jacobians. The image of π^* is an abelian subvariety of JS isogenous to $J(S/H)$. Thereby, the classical Poincaré's Reducibility theorem implies that there exists an abelian subvariety of JS, henceforth denoted by $\mathrm{Prym}(S \to S/H)$ and called the Prym variety associated to π, such that

$$JS \sim J(S/H) \times \mathrm{Prym}(S \to S/H),$$

The action of a finite group G on $\mathcal{S}$ induces a $\mathbb{Q}$-algebra homomorphism $\Xi : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(J\mathcal{S})$.
Let $W_1, \dots, W_r$ be the rational irreducible representations of G, and for each W_l let V_l be a complex irreducible representation of G associated to it. The equality

$$1 = e_1 + \cdots + e_r \quad \text{in} \quad \mathbb{Q}[G], \tag{1}$$

where e_l is uniquely determined central idempotent associated to W_l, yields a G-equivariant isogeny

$$J\mathcal{S} \sim A_1 \times \cdots \times A_r,$$

where A_l is the abelian subvariety of JS defined as $A_l := \Xi(ne_l)(J\mathcal{S})$ and $n \geqslant 1$ is chosen to satisfy $ne_l \in \mathbb{Z}[G]$. Moreover, there are idempotents $f_{l1}, \dots, f_{ln_l}$ such that

$$e_l = f_{l1} + \cdots + f_{ln_l} \tag{2}$$

where $n_l = d_l/s_l$ is the quotient of the degree d_l and the Schur index s_l of V_l. These idempotents provide n_l pairwise isogenous subvarieties of $J\mathcal{S}$. If we denote by B_l one of them for each l, then (1) and (2) provide the isogeny

$$J\mathcal{S} \sim B_1^{n_1} \times \cdots \times B_r^{n_r} \tag{3}$$

known as the group algebra decomposition of $J\mathcal{S}$ with respect to G.

If $H \leqslant G$ then we denote by $d_{V_l}^H$ the dimension of the vector subspace V_l^H of V_l of the elements fixed under H. The group algebra decomposition (3) induces the following isogenies.

1. The Jacobian variety JS_H of the quotient S_H decomposes as

$$JS_H \sim B_1^{n_1^H} \times \cdots \times B_r^{n_r^H} \quad \text{where} \quad n_l^H = d_{V_l}^H / s_{V_l}. \tag{4}$$

2. Let $H_1 \leqslant H_2$ be subgroups of G. The Prym variety associated to $S_{H_1} \to S_{H_2}$ decomposes as

$$\mathrm{Prym}(S_{H_1} \to S_{H_2}) \sim B_1^{n_1^{H_1,H_2}} \times \cdots \times B_r^{n_r^{H_1,H_2}} \quad \text{where} \quad n_l^{H_1,H_2} = n_l^{H_1} - n_l^{H_2}. \tag{5}$$

The previous induced isogenies are useful to provide decomposition of Jacobian varieties JS whose factors are isogenous to Jacobians of quotients of S and Pryms of intermediate coverings.
Assume that $(\gamma; m_1, \ldots, m_l)$ is the signature of the action of G on S and that this action is given by $\theta : \Delta \to G$, the dimension of B_i in (3) for $i \geqslant 2$ is given by

$$\dim B_i = k_{V_i}[d_{V_i}(\gamma - 1) + \frac{1}{2}\Sigma_{k=1}^l (d_{V_i} - d_{V_i}^{\langle\theta(x_k)\rangle})] \tag{6}$$

where k_{V_i} is the degree of the extension $\mathbb{Q} \leq L_{V_i}$ with L_{V_i} denoting a minimal field of definition for V_i. Note that the dimension of B_1 equals γ.
The decomposition of Jacobian varieties with group actions goes back to old works of Wirtinger, Schottky and Jung. For decompositions of Jacobians with respect to special groups Lange-Recillas, Lange-Rojas, Recillas-Rodriguez

Teichmüller and Moduli Spaces

Δ abstract Fuchsian group $\quad s(\Delta) = (h; m_1, \ldots, m_r)$

$\mathcal{T}_\Delta = \{\sigma : \Delta \to PSL(2, \mathbb{R}) \mid \sigma \mathit{injective},\ \sigma(\Delta)$ discrete $\}/PSL(2, \mathbb{R})$

Teichmüller space $\mathcal{T}_\Delta$ has a complex structure of dim $3h - 3 + r$, diffeomorphic to a ball of dim $6h - 6 + 2r$.

If Λ subgroup of Δ $(i : \Lambda \to \Delta) \Rightarrow i_* : \mathcal{T}_\Delta \to \mathcal{T}_\Lambda$ isometric embedding; $\quad i_*[\sigma] = [\sigma \circ i]$

Γ_g surface Fuchsian group $\quad \Gamma_g \leq \Delta \quad \mathcal{T}_\Delta \subset \mathcal{T}_{\Gamma_g} = \mathcal{T}_g$

G finite group $\quad \mathcal{T}_g^G = \{[\sigma] \in \mathcal{T}_g \mid g[\sigma] = [\sigma]\, \forall g \in G\} \neq \emptyset$
$\mathcal{T}_g^G$: surfaces with G as a group of automorphisms.

Mapping class group $M^+(\Delta) = Out(\Delta) = \frac{Diff(\mathbb{H}/\Delta)}{Diff_0(\mathbb{H}/\Delta)}$

$\Delta = \pi_1(\mathbb{H}/\Delta)$ as orbifold

$M^+(\Delta)$ acts properly discontinuously on $\mathcal{T}_\Delta$; $\mu \in M^+(\Delta)$, $[\sigma] \to \mu_*([\sigma]) = [\sigma \circ \mu^{-1}]$;

$$\mathcal{M}_\Delta = \mathcal{T}_\Delta / M^+(\Delta)$$

For $g = 1$ Schwarz

For $g = 2$ Bolza (1887, moduli of automorphic functions)
For hyperbolic surfaces (Teichmüller) Ahlfors, Bers, Harvey, Natanzon, Macbeath.

Deligne-Mumford Complection (going to ∞ in $\mathcal{M}_g$)

Curves whose singularities are ordinary double points (nodes), all of whose irreducible components isomorphic to $\mathbb{P}^1$ (or $\widehat{\mathbb{C}}$), meet the other irreducible components in at least 3 nodes: stable curves

$\widehat{\mathcal{M}}_g = \mathcal{M}_g \cup \{\text{stable curves}\}$

(deforming by variating the coeffcients or roots)

Geometrically: Riemann surfaces with a geodesic multicurve pinched to length 0

(deforming by variating the lengths of a system of curves)

Consider the complection $\widehat{\mathcal{B}}_g$ of $\mathcal{B}_g$ in $\widehat{\mathcal{M}}_g$

Surfaces with Non-Trivial Automorphisms

If Λ subgroup of Δ $(i : \Lambda \to \Delta) \Rightarrow i_* : \mathcal{T}_\Delta \to \mathcal{T}_\Lambda$ embedding
$\Gamma_g \leq \Delta \qquad \mathcal{T}_\Delta \subset \mathcal{T}_g$

G finite group $\qquad \mathcal{T}_g^G = \{[\sigma] \in \mathcal{T}_g \mid g[\sigma] = [\sigma]\, \forall g \in G\} \neq \emptyset$ Nielsen's Conjecture (proved 1980) $\quad \mathcal{T}_g^G$: surfaces with G as a group of automorphisms.

Marked surface $\sigma(X) \in \mathcal{T}_g$ and $\beta \in M_g^+$,

$$\begin{array}{ccc} \mathbb{H}/\Gamma_g = X & \stackrel{\sigma}{\to} & \sigma(X) \\ \downarrow & & \downarrow \\ \beta_*(X) & \stackrel{\sigma}{\to} & \sigma\beta(X) \end{array} \quad \text{biconformal}$$

$\beta[\sigma] = [\sigma] \qquad \Leftrightarrow \qquad \gamma \in PSL(2, \mathbb{R}), \quad \sigma(\Gamma_g) = \gamma^{-1}\sigma\beta(\Gamma_g)\gamma$

γ induces an automorphism of the RS $[\sigma(X)]$, $Stb_{\mathcal{M}_g}[\sigma] = Aut([\sigma(X)])$

Action (the monodromy): $\theta : \Delta \to Aut(X_g) = G$, $ker(\theta) = \Gamma_g$

Harvey 1971: $\mathcal{T}_g^G = \bigcup Im(i_*)$, for normal inclusions $i : \Gamma_g \to \Delta$ such that $G \cong \Delta/\Gamma_g$.

$\mu \in M^+(\Delta)$ one has $i^*(\mu_*) : i_*(\mathcal{T}_\Delta) \subset \mathcal{T}_g \to i_*(\mathcal{T}_\Delta) \subset \mathcal{T}_g$ given by
$$i^*(\mu_*)[\rho \circ i] = [\rho \circ i \circ \mu^{-1}].$$

For $g \geq 3$ the **branch locus** of the (orbifold-) universal covering $\mathcal{T}_g \to \mathcal{M}_g$ consists of the RS with non-trivial automorphisms

Equisymmetric Stratification

Two (surface) monodromies $\theta_1, \theta_2 : \Delta \to G$ topologically equiv. actions of G

$$\begin{array}{cccccc} & \Delta & \stackrel{\theta_1}{\to} & G & \\ \beta \in Aut(\Delta) & \downarrow & & \downarrow & w \in Aut(G) \\ & \Delta & \stackrel{\theta_2}{\to} & G & \end{array}$$

θ_1, θ_2 equiv under $\mathcal{B}(\Delta) \times Aut(G)$, $\mathcal{B}(\Delta)$ **braid group**

Broughton (1990): **Equisymmetric Stratification**

$\mathcal{M}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{ is } G\}$, teh conjugacy class of G in M_g^+

$\overline{\mathcal{M}}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{ contains } G\}$

$\overline{\mathcal{M}}_g^{G,\theta}$ connected, closed alg. var. of $\mathcal{M}_g$ with interior $\mathcal{M}_g^{G,\theta}$.

$\mathcal{M}_g^{G,\theta}$ empty iff $G \neq Aut(X_g)$ for any Riemann surface in $\overline{\mathcal{M}}_g^{G,\theta}$.

$$\mathcal{B}_g = \cup \overline{\mathcal{M}}_g^{G,\theta}$$

Costa-I (2008) $\mathcal{B}_g = \cup \overline{\mathcal{M}}_g^{C_p,\theta}$ (Cornalba 1987 and 2008)

Algorithm

Algorithm: to determine families of Riemann Surfaces X_g of genus g admitting actions of finite groups G of prescribed order: determine signatures of Fuchsian groups Δ (using Riemann-Hurwitz) and monodromies θ such that $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma_g$.

In the same way to show that a surface Y, belongs to the closure of the family above find supergroups G' of G and Λ of Δ such that the monodromy $\theta : \Delta \to G$ extends to a monodromy $\theta' : \Lambda \to G'$.

Recall that the different actions are given by the classes of epimorphisms

$$\begin{array}{ccc} \Gamma_g & & \langle 1_d \rangle \\ \downarrow & & \downarrow \\ \Delta & & G \\ \downarrow & & \downarrow \\ \Gamma & & G' \end{array} \qquad \beta \in Aut(\Delta) \quad \begin{array}{ccc} \Delta & \xrightarrow{\theta_1} & G \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\theta_2} & G \end{array} \quad w \in Aut(G)$$

Since we are interested not in actions but in families of RS we need ($Aut(X_g)$: **full automorphism group**)

Since actions $\theta : \Delta \to Aut(X_g)$, $\quad ker(\theta) = \Gamma_g$, we need

Singerman's list of non-maximal signatures.

A signature s is called **finitely maximal** if for any Fuchsian group Δ with $s(\Delta) = s$ and a group Δ' containing Δ we have $\dim \mathcal{T}_{\Delta'} < \dim \mathcal{T}_{\Delta}$

First Examples

All three families

- The equisymmetric family, González, in genera $g = (p-1)^2$, $p \geq 3$ with $4g + 4 + 8\sqrt{g}$ autom. and gr. $(C_p \times C_p) \rtimes (C_2 \times C_2)$.
- The equisymmetric family of dimension one whose surfaces have $4(g+1)$ automorphisms, gr. $G = D_{g+1} \times C_2$
- The equisymmetric family of dimension one whose surfaces have $4g$ automorphisms, gr. $G = D_{2g}$

are Riemann surfaces. In fact they are the Riemann sphere with three punctures: two punctures in the boundary of the moduli space and one puncture in the moduli space. The Riemann surface corresponding to the puncture in the moduli space is

- For González: one surface with $8p^2$ automorphisms and gr. $(C_p \times C_p) \rtimes D_4$, supporting one reflexive map of type $\{4, 2p\}$.
- For the surfaces with automorphism gr. $G = D_{g+1} \times C_2$: Accola-Maclachlan's curve $y^2 = x(x^{2g+2} - 1)$, supporting one reflexive map of type $\{4, 2g+2\}$.
- For the surfaces with automorphism gr. $G = D_{2g}$: Wiman's curve of type II $y^2 = x(x^{2g} - 1)$, supporting one reflexive map of type $\{4, 4g\}$.

Costa-I-Ying 2010, Bujalance-Costa-I 2017 (Jacobian studied by Reyes-Carocca 2018), Costa-I 2018

Other 1-parametric families

1. Let $g \geqslant 8$ such that $g-1$ is prime. There exists a compact Riemann surface of genus g with a group of automorphisms of order $6(g-1)$ if and only if $g \equiv 2 \bmod 3$. Moreover, in this case. The Riemann surfaces form a closed one-dimensional equisymmetric family $\bar{\mathcal{F}}_g$ of Riemann surfaces S with a group of automorphisms G isomorphic to $C_{g-1} \rtimes_6 C_6$. G acts with signature $(0; 2, 2, 3, 3)$. $\bar{\mathcal{F}}_g$ contains two Riemann surfaces X_1 and X_2 with a group of automorphisms G' of order $12(g-1)$ isomorphic to $(C_{g-1} \rtimes_6 C_6) \times C_2$, acting with signature $(0; 2, 6, 6)$, (X_1, X_2 founded by Belolipetski-Jones) I-Reyes 2020
2. Assume $g \equiv -1 \bmod 4$. There is a nodal, real equisymmetric uniparametric family $\mathcal{K}_g$ of genus 2 formed by (non-hyperelliptic) Riemann surfaces with automorphism group isomorphic to the central product of D_4 with the C_{g+1}, acting with signature $(0; +; [2, 2, 2, g+1])$. The family $\mathcal{K}_g$ is obtained as a covering of the Riemann Sphere with two punctures: one is Kulkarni curve $(y^{2g+2} = x(x-1)^{g-1}(x+1)^{g+2},)$ and the other is a nodal surface with topological type of two spheres joined by $g+1$ nodes. (Costa-I, Broughton-Costa-I 2022)
3. Consider the groups $G = C_q \rtimes C_3 = \left\langle s, t : s^q = t^3 = 1, t^2st = s^u \right\rangle$ where $q \equiv 1 \bmod 3$, q prime and with u satisfying $u^2 \equiv -u-1 \bmod q$. The Riemann surfaces of genus $\frac{3q-1}{2}$ with G as (full) automorphism group form an equisymmetric family of dim 1 which is a Riemann surface of genus 2 and (before compactification) six punctures. (Broughton-Costa-I 2023)

THE END