Schwarz–Pick Systems and Teichmüller spaces

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Throughout this talk, Δ denotes the open unit disk in the complex plane $\mathbb{C}$; so,

$$\Delta := \{z \in \mathbb{C} : |z| < 1\}.$$

The Poincaré metric on Δ is given by:

$$\rho_\Delta(z_1, z_2) = \tanh^{-1} \left| \frac{z_1 - z_2}{1 - z_1 \bar{z_2}} \right|$$

for $z_1, z_2 \in \Delta$.

If X and Y are complex manifolds, we shall denote the set of all holomorphic maps of X into Y by $\mathcal{O}(X, Y)$.

The term *Schwarz–Pick system* was first coined by L. A. Harris.

Definition 1. A Schwarz-Pick system is a functor, denoted by $X \mapsto d_X$, that assigns to each complex manifold X a pseudometric d_X so that the following conditions hold:
(i) The pseudometric assigned to Δ is the Poincaré metric, and
(ii) If X and Y are complex manifolds then

$$d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$$

if $x_1, x_2 \in X$ and $f \in \mathcal{O}(X, Y)$.

Because of conditions (i) and (ii) the sets $\mathcal{O}(\Delta, X)$ and $\mathcal{O}(X, \Delta)$ provide upper and lower bounds for d_X. These upper and lower bounds lead to the definitions of the Kobayashi and Carathéodory pseudometrics.

Definition 2. A Schwarz-Pick pseudometric on the complex manifold X is a pseudometric d such that

$$d(f(z), f(w)) \leq \rho_\Delta(z, w)$$

for all z and w in Δ and f in $\mathcal{O}(\Delta, X)$.

If $X \mapsto d_X$ is a Schwarz-Pick system, then d_X is obviously a Schwarz-Pick pseudometric on X for every complex manifold X.

Definition 3. The **Kobayashi pseudometric** K_X is the largest Schwarz-Pick pseudometric on the complex manifold X.

Let x and y be points of the complex manifold X. Since the Kobayashi pseudometrics form a Schwarz-Pick system, Definition 1 implies that

$$\rho_\Delta(f(x), f(y)) \leq K_X(x, y)$$

for all f in $\mathcal{O}(X, \Delta)$. Therefore, the number

$$C_X(x, y) = \sup\{\rho_\Delta(f(x), f(y)) : f \in \mathcal{O}(X, \Delta)\}$$

is finite and is bounded by $K_X(x, y)$.

Definition 4. We call C_X the **Carathéodory pseudometric** on the complex manifold X.

Some facts.

1. For the open unit disk Δ, we have:

$$K_\Delta(z, w) = C_\Delta(z, w) = \rho_\Delta(z, w)$$

for all z, w in Δ.

2. Let $B_r(a)$ be the open ball of radius r and center a in a complex Banach space X. Then

$$K_{B_r(a)}(a, x) = C_{B_r(a)}(a, x) = \tanh^{-1}\left(\frac{\|x - a\|}{r}\right)$$

for all x in $B_r(a)$.

Theorem A. (Earle and Mitra) Let X and Y be connected complex manifolds modelled on complex Banach spaces and let Φ be a holomorphic map of Y onto X. Suppose that for every f in $\mathcal{O}(\Delta, X)$ and every y in Y with $\Phi(y) = f(0)$ there is a function g in $\mathcal{O}(\Delta, Y)$ such that $g(0) = y$ and $f = \Phi \circ g$. Then

$$K_X(\Phi(y), x') = \inf\{K_Y(y, y') : y' \in Y \text{ and } \Phi(y') = x'\}$$

for all y in Y and x' in X.

The map $g : \Delta \to Y$ is called the **holomorphic lift** of the map $f : \Delta \to X$.

Teichmüller spaces are complex manifolds, with a natural metric, called the Teichmüller metric. The study of Kobayashi and Carathéodory metrics on Teichmüller spaces is an important topic. An important theorem of Royden states that the Teichmüller and Kobayashi metrics coincide for finite dimensional Teichmüller spaces. Royden's theorem was extended to all Teichmüller spaces by Gardiner. Subsequently, using holomorphic motions, an easy proof was given by Earle, Kra, and Krushkaĺ.

The question of Carathéodory metric on Teichmüller spaces was first studied by Earle. In that paper, using Bers embedding, Earle showed that the Carathéodory metric is complete on Teichmüller spaces. In that same paper, Earle asked the question whether the Carathéodory metric coincides with the Teichmüller metric on Teichmüller spaces. In an important paper, Marković (2018), proved that for any closed surface of genus $g \geq 2$, the answer is negative.

In a recent paper *Carathéodory metric on some generalized Teichmüller spaces*, by Xinlong Dong and Sudeb Mitra (Annales Fennici Mathematatici, **48**, 2023, 797–807), the authors sharpened an inequality of Earle's paper, and proved that, for a large class of Teichmüller spaces, that is, the *product Teichmüller space*, and the *Teichmüller space of a closed set in the Riemann sphere*, the Carathéodory metric is complete.

Let C_T and K_T denote the Carathéodory metrics on the Teichmüller space $Teich(X)$; here, X may be a hyperbolic Riemann surface, or X may be the disjoint union of hyperbolic Riemann surfaces X_i i.e. $X = \coprod_{i \in I} X_i$; and I is an index set, which may be finite, countable, or uncountable.

The crucial part of the Dong–Mitra paper was the following inequality:

Theorem B. (Dong and Mitra)
$\tanh C_T(x,y) \leq \tanh K_T(x,y) \leq 3 \tanh C_T(x,y)$ for all x, y in $Teich(X)$.

In what follows, we discuss some generalized Teichmüller spaces, and their metric properties. These Teichmüller spaces are intimately related with the study of holomorphic motions, and also have several applications in geometric function theory.

We will use the following notation: $\mathbb{C}$ for the complex plane, and $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for the Riemann sphere.

Definition 4. A complex-valued function $w = f(z)$ defined on a region Ω in $\mathbb{C}$ is called a quasiconformal mapping if it is a sense–preserving homeomorphism of Ω onto its image and its complex distributional derivatives

$$w_z = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big) \text{ and } w_{\overline{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)$$

are Lebesgue measurable locally square integrable functions on Ω that satisfy the inequality $|w_{\overline{z}}| \leq k|w_z|$ almost everywhere in Ω, for some real number k with $0 \leq k < 1$.

If $w = f(z)$ is a quasiconformal mapping defined on the region Ω then the function w_z is known to be nonzero almost everywhere on Ω. Therefore the function

$$\mu_f = \frac{w_{\overline{z}}}{w_z}$$

is a well–defined L^∞ function on Ω, called the *complex dilatation* or the *Beltrami coefficient* of f. The L^∞ norm of every Beltrami coefficient is less than one.

The positive number

$$K(f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}$$

is called the *dilatation* of f. We say that f is K-quasiconformal if f is a quasiconformal mapping and $K(f) \leq K$.

We call a homeomorphism of $\widehat{\mathbb{C}}$ *normalized* if it fixes the points 0, 1, and ∞.

Let $M(\mathbb{C})$ denote the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Then, for each μ in $M(\mathbb{C})$, there exists a unique normalized quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ onto itself that has Beltrami coefficient μ; we denote this quasiconformal map by w^μ. Furthermore, for every fixed $z \in \widehat{\mathbb{C}}$, the map $\mu \mapsto w^\mu(z)$ of $M(\mathbb{C})$ into $\widehat{\mathbb{C}}$ is holomorphic. The basepoint of $M(\mathbb{C})$ is the zero function.

Let E be a closed set in $\widehat{\mathbb{C}}$, containing 0, 1, and ∞.

Definition 6. Two normalized quasiconformal self-mappings f and g of $\widehat{\mathbb{C}}$ are said to be E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The *Teichmüller space* $T(E)$ is the set of all E-equivalence classes of normalized quasiconformal self-mappings of $\widehat{\mathbb{C}}$. The basepoint of $T(E)$ is the E-equivalence class of the identity map.

We can define the quotient map

$$P_E : M(\mathbb{C}) \to T(E)$$

by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ, written as $[w^\mu]_E$. Clearly, P_E maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$.

The Teichmüller space $T(E)$ is a simply–connected complex Banach manifold such that the projection map P_E from $M(\mathbb{C})$ to $T(E)$ is a holomorphic split submersion.

The Kobayashi metric on $M(\mathbb{C})$ is given by:

$$\rho_M(\mu, \nu) = \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \mu\bar{\nu}} \right\|_\infty$$

for all μ and ν in $M(\mathbb{C})$.

The Teichmüller metric $d_{T(E)}$ on $T(E)$ is given by:

$$d_{T(E)}(P_E(\mu), t) = \inf\{\rho_M(\mu, \nu) : \nu \in M(\mathbb{C}) \text{ and } P_E(\nu) = t\}$$

for all μ in $M(\mathbb{C})$ and t in $T(E)$.

Theorem C. (Earle and Mitra) Let f be any holomorphic map of Δ into $T(E)$, and let μ be any point in $M(\mathbb{C})$ such that $P_E(\mu) = f(0)$. There is a holomorphic map g from Δ to $M(\mathbb{C})$ such that $g(0) = \mu$ and $P_E \circ g = f$.

The following is a consequence of Theorems A and C.

Theorem D. (Earle and Mitra) The Teichmüller metric on $T(E)$ is the same as its Kobayashi metric.

Theorem E. (Dong and Mitra) The Carathéodory metric on $T(E)$ is complete.

An ongoing joint work with Arshiya Farhath. G (of the Ramanujan School of Mathematical Sciences, Pondicherry University, India), we are extending the above results to the Teichmüller space of a hyperbolic Riemann surface X, rel a closed set E in X; we denote this by $T(X,E)$.

Let X be a hyperbolic Riemann surface and let E be a closed set in E. As usual, let $M(X)$ is the open unit ball of the complex Banach space of Beltrami forms on X, of finite norm, denoted by $Belt(X)$. Let $QC_0(X,E)$ denote the set of all quasiconformal maps of X onto itself, that are isotopic to the identity, rel $E \cup \partial X$, where ∂X is the ideal boundary of X.

Then, the Teichmüller space $T(X,E)$ is defined as:

$$T(X,E) = M(X)/QC_0(X,E).$$

We are studying the Kobayashi and Carathéodory metrics on $T(X,E)$.

THANK YOU FOR YOUR ATTENTION!

Appendix I.

Let a Riemann surface X be obtained from a compact Riemann surface of genus g by removing n (≥ 0) points. If $2g + n - 2$ is positive, the Teichmüller space $Teich(g, n)$is the set of equivalence classes of quasiconformal (qc) mappings of X onto Riemann surfaces Y. Here two such mappings $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are equivalent if there is a conformal map $h : Y_1 \to Y_2$ such that the qc map $f_2^{-1} \circ h \circ f_1$ of X onto itself is isotopic to the identity.

The Teichmüller metric on $T(g, n)$ is defined by setting

$$d([f_1], [f_2]) = \frac{1}{2} \log K$$

where K is the smallest number (≥ 1) such that there is a K-qc mapping $f : f_1(X) \to f_2(X)$ isotopic to the map $f_2 \circ f_1^{-1}$.

Appendix II.

Let I be an index set. For full generality, in this section we will assume that I is uncountable. For each i in the index set I, let X_i be a hyperbolic Riemann surface. Let X be the disjoint union $\coprod_{i \in I} X_i$. We introduce the following important Banach spaces:

By definition, a Beltrami form on X is a tensor μ whose restriction to each X_i is a bounded measurable Beltrami form μ_i on X_i with L^∞ norm less than some finite constant independent of i in I. We define

$$\|\mu\| = \sup\{\|\mu_i\|_\infty : i \in I\}.$$

We denote the Banach space of Beltrami forms on X by $Belt(X)$ and we denote the open unit ball of $Belt(X)$ by $M(X)$. The basepoint of $M(X)$ is its center 0.

Let $\pi : \Delta \to R$ be a holomorphic universal covering of the hyperbolic Riemann surface R. Every holomorphic quadratic differential ψ on R lifts to a holomorphic quadratic differential $\widetilde{\psi}(z)dz^2$ on Δ. We say that ψ is bounded if its Nehari norm

$$\|\psi\|_N = \sup\{\|\widetilde{\psi}(z)\|(1 - |z|^2)^2 : z \in \Delta\}$$

is finite.

For each i in I let X_i^* be the conjugate Riemann surface of X_i, and let X^* be the disjoint union of the X_i^*. Let ψ be a holomorphic quadratic differential on X^*. We say that ψ is bounded if its restriction ψ_i to X_i^* is a bounded holomorphic quadratic differential for each i and its Nehari norm

$$\|\psi\|_N = \sup\{\|\psi_i\|_N : i \in I\}$$

is finite. We denote the complex Banach space of bounded holomorphic quadratic differentials on X^* by $B(X^*)$.

For each $i \in I$, let $Teich(X_i)$ be the Teichmüller space of the Riemann surface X_i, let 0_i be the basepoint of $Teich(X_i)$ and let d_i be the Teichmüller metric on $Teich(X_i)$. By definition, the product Teichmüller space $Teich(X)$ is the set of functions t on I such that $t(i)$ is in $Teich(X_i)$ for each i and the set of numbers $\{d_i(0_i, t(i)) : i \in I\}$ is bounded. As usual, we shall write t_i for $t(i)$. The basepoint of $Teich(X)$ is the function t such that $t_i = 0_i$ for each i; we shall denote it by 0_X.

The Teichmüller metric on $Teich(X)$ is defined by

$$d_T(s,t) = \sup\{d_i(s_i, t_i) : i \in I\},$$

for s and t in $Teich(X)$. Since each metric d_i is complete, the metric d_T on $Teich(X)$ is also complete.

Theorem. The Teichmüller metric on $Teich(X)$ is the same as its Kobayashi metric. The Carathéodory metric on $Teich(X)$ is complete.