Sufficiency of the Riemann-Hurwitz Formula for the Existence of a Group Action

A. Wootton (joint work with S. Bozlee)

Department of Mathematics
University of Portland,
Portland, Oregon 97203
wootton@up.edu

October 1, 2015

Overview

Introduction

The underlying question behind much of my work is the following:

Question

For a fixed genus $\sigma \geq 2$, can we describe all the possible finite group actions on a compact Riemann surface of genus $\sigma \geq 2$?

Introduction

The underlying question behind much of my work is the following:

Question

For a fixed genus $\sigma \geq 2$, can we describe all the possible finite group actions on a compact Riemann surface of genus $\sigma \geq 2$?

- Current results include:

Introduction

The underlying question behind much of my work is the following:

Question

For a fixed genus $\sigma \geq 2$, can we describe all the possible finite group actions on a compact Riemann surface of genus $\sigma \geq 2$?

- Current results include:
  - Classification for "small" genera (Condor, Breuer, Broughton)

Introduction

The underlying question behind much of my work is the following:

Question

For a fixed genus $\sigma \geq 2$, can we describe all the possible finite group actions on a compact Riemann surface of genus $\sigma \geq 2$?

- Current results include:
  - Classification for "small" genera (Condor, Breuer, Broughton)
  - Classification for certain special families (Harvey, Breuer)

Introduction

The underlying question behind much of my work is the following:

Question

For a fixed genus $\sigma \geq 2$, can we describe all the possible finite group actions on a compact Riemann surface of genus $\sigma \geq 2$?

- Current results include:
  - Classification for "small" genera (Condor, Breuer, Broughton)
  - Classification for certain special families (Harvey, Breuer)
- Not too much else is known as this is a hard problem!!!

Introduction

The underlying question behind much of my work is the following:

Question

For a fixed genus $\sigma \geq 2$, can we describe all the possible finite group actions on a compact Riemann surface of genus $\sigma \geq 2$?

- Current results include:
  - Classification for "small" genera (Condor, Breuer, Broughton)
  - Classification for certain special families (Harvey, Breuer)
- Not too much else is known as this is a hard problem!!!

What makes it difficult?

To prove the existence of a group action on a surface X of genus σ, there are two conditions that need verifying:

1. A number theoretic condition (the Riemann-Hurwitz formula)

What makes it difficult?

To prove the existence of a group action on a surface X of genus σ, there are two conditions that need verifying:

1. A number theoretic condition (the Riemann-Hurwitz formula)
2. A group theoretic condition (the existence of a generating vector)

What makes it difficult?

To prove the existence of a group action on a surface X of genus σ, there are two conditions that need verifying:

1. A number theoretic condition (the Riemann-Hurwitz formula)
2. A group theoretic condition (the existence of a generating vector)

The difficulty comes mainly in the second step. Naively speaking, as genus increases, there are more potential groups, and these groups have increasingly complicated structures.

What makes it difficult?

To prove the existence of a group action on a surface X of genus σ, there are two conditions that need verifying:

1. A number theoretic condition (the Riemann-Hurwitz formula)
2. A group theoretic condition (the existence of a generating vector)

The difficulty comes mainly in the second step. Naively speaking, as genus increases, there are more potential groups, and these groups have increasingly complicated structures.

A different approach

Rather than try to solve the classification of group actions in one step, consider the two conditions separately, starting with the more amenable problem. Specifically:

1. Try to completely understand the number theoretic condition first (the "easy" step)

A different approach

Rather than try to solve the classification of group actions in one step, consider the two conditions separately, starting with the more amenable problem. Specifically:

1. Try to completely understand the number theoretic condition first (the “easy” step)
2. Consider the group theoretic condition afterward (the “difficult” step)

A different approach

Rather than try to solve the classification of group actions in one step, consider the two conditions separately, starting with the more amenable problem. Specifically:

1. Try to completely understand the number theoretic condition first (the “easy” step)
2. Consider the group theoretic condition afterward (the “difficult” step)

This leads to my primary topic of discussion:

A different approach

Rather than try to solve the classification of group actions in one step, consider the two conditions separately, starting with the more amenable problem. Specifically:

1. Try to completely understand the number theoretic condition first (the "easy" step)
2. Consider the group theoretic condition afterward (the "difficult" step)

This leads to my primary topic of discussion:

Question

Is there a big difference between the number of things satisfying the "easy" condition, but not the "hard" condition?

A different approach

Rather than try to solve the classification of group actions in one step, consider the two conditions separately, starting with the more amenable problem. Specifically:

1. Try to completely understand the number theoretic condition first (the "easy" step)
2. Consider the group theoretic condition afterward (the "difficult" step)

This leads to my primary topic of discussion:

Question

Is there a big difference between the number of things satisfying the "easy" condition, but not the "hard" condition?

In order to answer this question, we must first formalize things.

A different approach

Rather than try to solve the classification of group actions in one step, consider the two conditions separately, starting with the more amenable problem. Specifically:

1. Try to completely understand the number theoretic condition first (the "easy" step)
2. Consider the group theoretic condition afterward (the "difficult" step)

This leads to my primary topic of discussion:

Question

Is there a big difference between the number of things satisfying the "easy" condition, but not the "hard" condition?

In order to answer this question, we must first formalize things.

Suppose that a finite group G acts on a surface X of genus σ.

Definition

We say that G has signature $(h; m_1, \ldots, m_r)$, $m_1 \leq m_2 \leq \cdots \leq m_r$ if the following are true:

Suppose that a finite group G acts on a surface X of genus σ.

Definition

We say that G has signature $(h; m_1, \ldots, m_r)$, $m_1 \leq m_2 \leq \cdots \leq m_r$ if the following are true:

1. The quotient space X/G has genus h.

Suppose that a finite group G acts on a surface X of genus σ.

Definition

We say that G has signature $(h; m_1, \ldots, m_r)$, $m_1 \leq m_2 \leq \cdots \leq m_r$ if the following are true:

1. The quotient space X/G has genus h.
2. The quotient map $\pi\colon X \to X/G$ is branched over r points with branching orders $m_1, \ldots, m_r$.

Suppose that a finite group G acts on a surface X of genus σ.

Definition

We say that G has signature $(h; m_1, \ldots, m_r)$, $m_1 \leq m_2 \leq \cdots \leq m_r$ if the following are true:

1. The quotient space X/G has genus h.
2. The quotient map $\pi \colon X \to X/G$ is branched over r points with branching orders $m_1, \ldots, m_r$.

- We call the numbers $m_1, \ldots, m_r$ the *periods* of the signature.

Suppose that a finite group G acts on a surface X of genus σ.

Definition

We say that G has signature $(h; m_1, \ldots, m_r)$, $m_1 \leq m_2 \leq \cdots \leq m_r$ if the following are true:

1. The quotient space X/G has genus h.
2. The quotient map $\pi \colon X \to X/G$ is branched over r points with branching orders $m_1, \ldots, m_r$.

- We call the numbers $m_1, \ldots, m_r$ the *periods* of the signature.

Suppose that G is a finite group and $\mathcal{S} = (h; m_1, \ldots, m_r)$ is a signature.

Definition

We say the vector $\mathcal{V} = (a_1, b_1, a_2, b_2, \ldots, a_h, b_h, g_1, \ldots, g_r)$ of elements of G is an $\mathcal{S}$-generating vector for G if the following hold:

Suppose that G is a finite group and $\mathcal{S} = (h; m_1, \ldots, m_r)$ is a signature.

Definition

We say the vector $\mathcal{V} = (a_1, b_1, a_2, b_2, \ldots, a_h, b_h, g_1, \ldots, g_r)$ of elements of G is an $\mathcal{S}$-generating vector for G if the following hold:

1. $O(g_i) = m_i$ (where O denotes order).

Suppose that G is a finite group and $\mathcal{S} = (h; m_1, \ldots, m_r)$ is a signature.

Definition

We say the vector $\mathcal{V} = (a_1, b_1, a_2, b_2, \ldots, a_h, b_h, g_1, \ldots, g_r)$ of elements of G is an $\mathcal{S}$-generating vector for G if the following hold:

1. $O(g_i) = m_i$ (where O denotes order).
2. $G = \langle a_1, b_1, a_2, b_2, \ldots, a_h, b_h, g_1, \ldots, g_r \rangle$.

Suppose that G is a finite group and $\mathcal{S} = (h; m_1, \ldots, m_r)$ is a signature.

Definition

We say the vector $\mathcal{V} = (a_1, b_1, a_2, b_2, \ldots, a_h, b_h, g_1, \ldots, g_r)$ of elements of G is an $\mathcal{S}$-generating vector for G if the following hold:

1. $O(g_i) = m_i$ (where O denotes order).
2. $G = \langle a_1, b_1, a_2, b_2, \ldots, a_h, b_h, g_1, \ldots, g_r \rangle$.
3. $\Pi_{i=1}^{h}[a_i, b_i]\Pi_{j=1}^{r} g_j = e$, the identity in G where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

Suppose that G is a finite group and $\mathcal{S} = (h; m_1, \ldots, m_r)$ is a signature.

Definition

We say the vector $\mathcal{V} = (a_1, b_1, a_2, b_2, \ldots, a_h, b_h, g_1, \ldots, g_r)$ of elements of G is an $\mathcal{S}$-generating vector for G if the following hold:

1. $O(g_i) = m_i$ (where O denotes order).
2. $G = \langle a_1, b_1, a_2, b_2, \ldots, a_h, b_h, g_1, \ldots, g_r \rangle$.
3. $\Pi_{i=1}^{h}[a_i, b_i]\Pi_{j=1}^{r} g_j = e$, the identity in G where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

The following are the two conditions necessary for the existence of a group action on a surface of genus σ:

Theorem

A group G acts on a surface X of genus σ with signature $\mathcal{S} = (h; m_1, \ldots, m_r)$ if and only if the following hold:

The following are the two conditions necessary for the existence of a group action on a surface of genus σ:

Theorem

A group G acts on a surface X of genus σ with signature $\mathcal{S} = (h; m_1, \ldots, m_r)$ if and only if the following hold:

1. *The Riemann Hurwitz formula holds:*

$$\sigma - 1 = |G|(h-1) + \frac{|G|}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

The following are the two conditions necessary for the existence of a group action on a surface of genus σ:

Theorem

A group G acts on a surface X of genus σ with signature $\mathcal{S} = (h; m_1, \ldots, m_r)$ if and only if the following hold:

1. *The Riemann Hurwitz formula holds:*

$$\sigma - 1 = |G|(h-1) + \frac{|G|}{2} \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right).$$

2. *G admits an $\mathcal{S}$-generating vector.*

The following are the two conditions necessary for the existence of a group action on a surface of genus σ:

Theorem

A group G acts on a surface X of genus σ with signature $\mathcal{S} = (h; m_1, \ldots, m_r)$ if and only if the following hold:

1. *The Riemann Hurwitz formula holds:*

$$\sigma - 1 = |G|(h-1) + \frac{|G|}{2} \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right).$$

2. *G admits an $\mathcal{S}$-generating vector.*

Reformulating the Question

We can now state our problem more formally:

Question

For a given σ and G, how many signatures satisfy the Riemann-Hurwitz formula but do not have generating vectors?

Reformulating the Question

We can now state our problem more formally:

Question

For a given σ and G, how many signatures satisfy the Riemann-Hurwitz formula but do not have generating vectors?

Potential Signatures

For a given G and σ we define two different sets of signatures.

Definition

We define $\mathcal{A}_\sigma(G)$ to be the set of signatures for which there exists an action of G on a surface of genus σ with that signature. We call these *actual signatures*.

Potential Signatures

For a given G and σ we define two different sets of signatures.

Definition

We define $\mathcal{A}_\sigma(G)$ to be the set of signatures for which there exists an action of G on a surface of genus σ with that signature. We call these *actual signatures*.

Definition

We define $\mathcal{P}_\sigma(G)$ to be the set of signatures which satisfy the following:

Potential Signatures

For a given G and σ we define two different sets of signatures.

Definition

We define $\mathcal{A}_\sigma(G)$ to be the set of signatures for which there exists an action of G on a surface of genus σ with that signature. We call these *actual signatures*.

Definition

We define $\mathcal{P}_\sigma(G)$ to be the set of signatures which satisfy the following:

1. Each period is the order of an element of G

Potential Signatures

For a given G and σ we define two different sets of signatures.

Definition

We define $\mathcal{A}_\sigma(G)$ to be the set of signatures for which there exists an action of G on a surface of genus σ with that signature. We call these *actual signatures*.

Definition

We define $\mathcal{P}_\sigma(G)$ to be the set of signatures which satisfy the following:

1. Each period is the order of an element of G
2. They satisfy the Riemann-Hurwitz formula for genus σ.

Potential Signatures

For a given G and σ we define two different sets of signatures.

Definition

We define $\mathcal{A}_\sigma(G)$ to be the set of signatures for which there exists an action of G on a surface of genus σ with that signature. We call these *actual signatures*.

Definition

We define $\mathcal{P}_\sigma(G)$ to be the set of signatures which satisfy the following:

1. Each period is the order of an element of G
2. They satisfy the Riemann-Hurwitz formula for genus σ.

We call these *potential* signatures.

Potential Signatures

For a given G and σ we define two different sets of signatures.

Definition

We define $\mathcal{A}_\sigma(G)$ to be the set of signatures for which there exists an action of G on a surface of genus σ with that signature. We call these *actual signatures*.

Definition

We define $\mathcal{P}_\sigma(G)$ to be the set of signatures which satisfy the following:

1. Each period is the order of an element of G
2. They satisfy the Riemann-Hurwitz formula for genus σ.

We call these *potential* signatures.

Question

So our question is: what is the size of $\mathcal{A}_\sigma(G)$ relative to $\mathcal{P}_\sigma(G)$?

Potential Signatures

For a given G and σ we define two different sets of signatures.

Definition

We define $\mathcal{A}_\sigma(G)$ to be the set of signatures for which there exists an action of G on a surface of genus σ with that signature. We call these *actual signatures*.

Definition

We define $\mathcal{P}_\sigma(G)$ to be the set of signatures which satisfy the following:

1. Each period is the order of an element of G
2. They satisfy the Riemann-Hurwitz formula for genus σ.

We call these *potential* signatures.

Question

So our question is: what is the size of $\mathcal{A}_\sigma(G)$ relative to $\mathcal{P}_\sigma(G)$?

Examples

Example

$$\mathcal{P}_6(C_6) = \left\{ \begin{array}{ccc} (0;3,3,3,6,6) & (0;2,3,6,6,6) & (0;2,2,3,3,3,3) \\ (0;2,2,2,3,3,6) & (0;2,2,2,2,6,6) & (0;2,2,2,2,2,2,3) \\ (1;6,6) & (1;2,2,3) & \end{array} \right\}$$

$$\mathcal{A}_6(C_6) = \left\{ \begin{array}{ccc} (0;3,3,3,6,6) & (0;2,3,6,6,6) & (0;2,2,3,3,3,3) \\ (0;2,2,2,3,3,6) & (0;2,2,2,2,6,6) & (1;6,6) \end{array} \right\}$$

Examples

Example

$$\mathcal{P}_6(C_6) = \left\{ \begin{array}{ccc} (0;3,3,3,6,6) & (0;2,3,6,6,6) & (0;2,2,3,3,3,3) \\ (0;2,2,2,3,3,6) & (0;2,2,2,2,6,6) & (0;2,2,2,2,2,2,3) \\ (1;6,6) & (1;2,2,3) & \end{array} \right\}$$

$$\mathcal{A}_6(C_6) = \left\{ \begin{array}{ccc} (0;3,3,3,6,6) & (0;2,3,6,6,6) & (0;2,2,3,3,3,3) \\ (0;2,2,2,3,3,6) & (0;2,2,2,2,6,6) & (1;6,6) \end{array} \right\}$$

So $|\mathcal{A}_6(C_6)| = 6$ and $|\mathcal{P}_6(C_6)| = 8$

Examples

Example

$$\mathcal{P}_6(C_6) = \left\{ \begin{array}{ccc} (0;3,3,3,6,6) & (0;2,3,6,6,6) & (0;2,2,3,3,3,3) \\ (0;2,2,2,3,3,6) & (0;2,2,2,2,6,6) & (0;2,2,2,2,2,2,3) \\ (1;6,6) & (1;2,2,3) & \end{array} \right\}$$

$$\mathcal{A}_6(C_6) = \left\{ \begin{array}{ccc} (0;3,3,3,6,6) & (0;2,3,6,6,6) & (0;2,2,3,3,3,3) \\ (0;2,2,2,3,3,6) & (0;2,2,2,2,6,6) & (1;6,6) \end{array} \right\}$$

So $|\mathcal{A}_6(C_6)| = 6$ and $|\mathcal{P}_6(C_6)| = 8$

Example

In genus $\sigma = 2$, there are 33 potential signatures and 19 actual signatures.

Examples

Example

$$\mathcal{P}_6(C_6) = \left\{ \begin{array}{ccc} (0;3,3,3,6,6) & (0;2,3,6,6,6) & (0;2,2,3,3,3,3) \\ (0;2,2,2,3,3,6) & (0;2,2,2,2,6,6) & (0;2,2,2,2,2,2,3) \\ (1;6,6) & (1;2,2,3) & \end{array} \right\}$$

$$\mathcal{A}_6(C_6) = \left\{ \begin{array}{ccc} (0;3,3,3,6,6) & (0;2,3,6,6,6) & (0;2,2,3,3,3,3) \\ (0;2,2,2,3,3,6) & (0;2,2,2,2,6,6) & (1;6,6) \end{array} \right\}$$

So $|\mathcal{A}_6(C_6)| = 6$ and $|\mathcal{P}_6(C_6)| = 8$

Example

In genus $\sigma = 2$, there are 33 potential signatures and 19 actual signatures.

An Asymptotic Bound for $\mathcal{A}_\sigma(G)$

For a given finite group G, we define the *order set* of G to be:

$$\mathcal{O}(G) = \{|x| : x \in G \backslash \langle e \rangle\} = \{n_1, \ldots, n_r\}$$

Maclachlan and Miller gave the following asymptotic bound on actual signatures:

An Asymptotic Bound for $\mathcal{A}_\sigma(G)$

For a given finite group G, we define the *order set* of G to be:

$$\mathcal{O}(G) = \{|x| : x \in G \backslash \langle e \rangle\} = \{n_1, \ldots, n_r\}$$

Maclachlan and Miller gave the following asymptotic bound on actual signatures:

$$|\mathcal{A}_\sigma(G)| \sim \left(\frac{A 2^{r-1}}{|G| \exp(G) r! \prod_{i=1}^{r} (1 - 1/n_i)} \right) \sigma^r$$

- $\exp(G)$ denotes the exponent of G
- $A = 2$ if $|G|$ is odd and $A = 1$ else

An Asymptotic Bound for $\mathcal{A}_\sigma(G)$

For a given finite group G, we define the *order set* of G to be:

$$\mathcal{O}(G) = \{|x| : x \in G \backslash \langle e \rangle\} = \{n_1, \ldots, n_r\}$$

Maclachlan and Miller gave the following asymptotic bound on actual signatures:

$$|\mathcal{A}_\sigma(G)| \sim \left(\frac{A2^{r-1}}{|G| \exp(G) r! \prod_{i=1}^{r} (1 - 1/n_i)} \right) \sigma^r$$

- $\exp(G)$ denotes the exponent of G
- $A = 2$ if $|G|$ is odd and $A = 1$ else

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

We determine an asymptotic bound on $\mathcal{P}_\sigma(G)$ as follows:

- Any potential signature can be written $(h; [n_1, t_1], [n_2, t_2], \ldots, [n_r, t_r])$ where the pair $[n_i, t_i]$ denotes t_i copies of n_i (note that $t_i \geq 0$).

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

We determine an asymptotic bound on $\mathcal{P}_\sigma(G)$ as follows:

- Any potential signature can be written $(h; [n_1, t_1], [n_2, t_2], \ldots, [n_r, t_r])$ where the pair $[n_i, t_i]$ denotes t_i copies of n_i (note that $t_i \geq 0$).
- In this form, the Riemann-Hurwitz formula becomes

$$2\sigma - 2 + 2|G| = 2|G|h + \sum_{i=1}^{r} \frac{|G|}{n_i}(n_i - 1)t_i$$

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

We determine an asymptotic bound on $\mathcal{P}_\sigma(G)$ as follows:

- Any potential signature can be written $(h; [n_1, t_1], [n_2, t_2], \ldots, [n_r, t_r])$ where the pair $[n_i, t_i]$ denotes t_i copies of n_i (note that $t_i \geq 0$).
- In this form, the Riemann-Hurwitz formula becomes

$$2\sigma - 2 + 2|G| = 2|G|h + \sum_{i=1}^{r} \frac{|G|}{n_i}(n_i - 1)t_i$$

- $|\mathcal{P}_\sigma(G)|$ is the number of tuples of non-negative integers $(h, t_1, \ldots, t_r)$ satisfying this formula.

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

We determine an asymptotic bound on $\mathcal{P}_\sigma(G)$ as follows:

- Any potential signature can be written $(h; [n_1, t_1], [n_2, t_2], \ldots, [n_r, t_r])$ where the pair $[n_i, t_i]$ denotes t_i copies of n_i (note that $t_i \geq 0$).
- In this form, the Riemann-Hurwitz formula becomes

$$2\sigma - 2 + 2|G| = 2|G|h + \sum_{i=1}^{r} \frac{|G|}{n_i}(n_i - 1)t_i$$

- $|\mathcal{P}_\sigma(G)|$ is the number of tuples of non-negative integers $(h, t_1, \ldots, t_r)$ satisfying this formula.

An Example

Example

When $G = C_6$, we have:

- $\mathcal{O}(G) = \{2, 3, 6\}$.

An Example

Example

When $G = C_6$, we have:

- $\mathcal{O}(G) = \{2, 3, 6\}$.

Therefore, for $\mathcal{P}_6(C_6)$, we are looking for all solutions (h, t_1, t_2, t_3) to:

An Example

Example

When $G = C_6$, we have:

- $\mathcal{O}(G) = \{2, 3, 6\}$.

Therefore, for $\mathcal{P}_6(C_6)$, we are looking for all solutions (h, t_1, t_2, t_3) to:

$$22 = 12h + 3t_1 + 4t_2 + 5t_3$$

An Example

Example

When $G = C_6$, we have:

- $\mathcal{O}(G) = \{2, 3, 6\}$.

Therefore, for $\mathcal{P}_6(C_6)$, we are looking for all solutions (h, t_1, t_2, t_3) to:

$$22 = 12h + 3t_1 + 4t_2 + 5t_3$$

Since $|G|$ and the n_i are fixed, we can think of this problem as:

An Example

Example

When $G = C_6$, we have:

- $\mathcal{O}(G) = \{2, 3, 6\}$.

Therefore, for $\mathcal{P}_6(C_6)$, we are looking for all solutions (h, t_1, t_2, t_3) to:

$$22 = 12h + 3t_1 + 4t_2 + 5t_3$$

Since $|G|$ and the n_i are fixed, we can think of this problem as:

- For a fixed K and $a, a_1, \ldots a_r$, find all non-negative integer solutions $(h, t_1, \ldots, t_r)$ to the equation:

$$K = ah + a_1 t_1 + a_2 t_2 + \ldots a_r t_r$$

An Example

Example

When $G = C_6$, we have:

- $\mathcal{O}(G) = \{2, 3, 6\}$.

Therefore, for $\mathcal{P}_6(C_6)$, we are looking for all solutions (h, t_1, t_2, t_3) to:

$$22 = 12h + 3t_1 + 4t_2 + 5t_3$$

Since $|G|$ and the n_i are fixed, we can think of this problem as:

- For a fixed K and a, $a_1, \ldots a_r$, find all non-negative integer solutions $(h, t_1, \ldots, t_r)$ to the equation:

$$K = ah + a_1 t_1 + a_2 t_2 + \ldots a_r t_r$$

Schur's Theorem

As it turns out, we have a result to solve precisely this problem:

Theorem

(Schur's Theorem) If $\{a_1, \ldots, a_n\}$ *is a set of integers such that* $\gcd(a_1, \ldots, a_n) = 1$ *and* S_x *is the number of different tuples of non-negative integers* $(c_1, \ldots, c_n)$ *such that* $x = c_1 a_1 + \cdots + c_n a_n$ *then*

$$S_x \sim \frac{x^{n-1}}{(n-1)! a_1 \ldots a_n}.$$

Schur's Theorem

As it turns out, we have a result to solve precisely this problem:

Theorem

(Schur's Theorem) If $\{a_1, \ldots, a_n\}$ *is a set of integers such that* $\gcd(a_1, \ldots, a_n) = 1$ *and* S_x *is the number of different tuples of non-negative integers* $(c_1, \ldots, c_n)$ *such that* $x = c_1 a_1 + \cdots + c_n a_n$ *then*

$$S_x \sim \frac{x^{n-1}}{(n-1)! a_1 \ldots a_n}.$$

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

- We cannot directly apply Schur's theorem as the coefficients, $2|G|$ and $\frac{|G|}{n_1}(n_1 - 1), \ldots, \frac{|G|}{n_r}(n_r - 1)$ may not be relatively prime.
- However, letting $A = 2$ if $|G|$ is odd and $A = 1$ if $|G|$ is even, we can show:

$$\gcd\left(2|G|, \frac{|G|}{n_1}(n_1 - 1), \frac{|G|}{n_2}(n_2 - 1), \ldots, \frac{|G|}{n_r}(n_r - 1)\right) = \frac{A|G|}{\exp(G)}$$

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

- We cannot directly apply Schur's theorem as the coefficients, $2|G|$ and $\frac{|G|}{n_1}(n_1-1),\ldots,\frac{|G|}{n_r}(n_r-1)$ may not be relatively prime.
- However, letting $A=2$ if $|G|$ is odd and $A=1$ if $|G|$ is even, we can show:

$$\gcd\left(2|G|,\frac{|G|}{n_1}(n_1-1),\frac{|G|}{n_2}(n_2-1),\ldots,\frac{|G|}{n_r}(n_r-1)\right)=\frac{A|G|}{\exp(G)}$$

- Rewriting the Riemann-Hurwitz formula:

$$\frac{2g-2+2|G|}{A|G|/\exp(G)}=\left(\frac{2|G|}{A|G|/\exp(G)}\right)h+\sum_{i=1}^{r}\left(\frac{|G|(n_i-1)/n_i}{A|G|/\exp(G)}\right)t_i$$

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

- We cannot directly apply Schur's theorem as the coefficients, $2|G|$ and $\frac{|G|}{n_1}(n_1 - 1), \ldots, \frac{|G|}{n_r}(n_r - 1)$ may not be relatively prime.
- However, letting $A = 2$ if $|G|$ is odd and $A = 1$ if $|G|$ is even, we can show:

$$\gcd\left(2|G|, \frac{|G|}{n_1}(n_1 - 1), \frac{|G|}{n_2}(n_2 - 1), \ldots, \frac{|G|}{n_r}(n_r - 1)\right) = \frac{A|G|}{\exp(G)}$$

- Rewriting the Riemann-Hurwitz formula:

$$\frac{2g - 2 + 2|G|}{A|G|/\exp(G)} = \left(\frac{2|G|}{A|G|/\exp(G)}\right) h + \sum_{i=1}^{r} \left(\frac{|G|(n_i - 1)/n_i}{A|G|/\exp(G)}\right) t_i$$

- The number of solutions to this equation will be the same as the number to the original equation. In this case however, the coefficients are now relatively prime, so we may use Schur's Theorem

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

- We cannot directly apply Schur's theorem as the coefficients, $2|G|$ and $\frac{|G|}{n_1}(n_1-1),\ldots,\frac{|G|}{n_r}(n_r-1)$ may not be relatively prime.
- However, letting $A=2$ if $|G|$ is odd and $A=1$ if $|G|$ is even, we can show:

$$\gcd\left(2|G|, \frac{|G|}{n_1}(n_1-1), \frac{|G|}{n_2}(n_2-1), \ldots, \frac{|G|}{n_r}(n_r-1)\right) = \frac{A|G|}{\exp(G)}$$

- Rewriting the Riemann-Hurwitz formula:

$$\frac{2g-2+2|G|}{A|G|/\exp(G)} = \left(\frac{2|G|}{A|G|/\exp(G)}\right)h + \sum_{i=1}^{r}\left(\frac{|G|(n_i-1)/n_i}{A|G|/\exp(G)}\right)t_i$$

- The number of solutions to this equation will be the same as the number to the original equation. In this case however, the coefficients are now relatively prime, so we may use Schur's Theorem

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

Applying Schur's Theorem, we get:

$$\mathcal{P}_\sigma(G) \sim \frac{\left(\frac{2\sigma-2+2|G|}{A|G|/\exp(G)}\right)^r}{r!\left[\frac{2|G|}{A|G|/\exp(G)}\right]\prod_{i=1}^{r}\left[\frac{|G|(1-1/n_i)}{(A|G|/\exp(G)}\right]}$$

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

Applying Schur's Theorem, we get:

$$\mathcal{P}_\sigma(G) \sim \frac{\left(\frac{2\sigma-2+2|G|}{A|G|/\exp(G)}\right)^r}{r!\left[\frac{2|G|}{A|G|/\exp(G)}\right]\prod_{i=1}^{r}\left[\frac{|G|(1-1/n_i)}{(A|G|/\exp(G)}\right]}$$

$$\sim \frac{2^r A|G|/\exp(G)}{r!2|G|^{r+1}\prod_{i=1}^{r}(1-1/n_i)}\sigma^r$$

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

Applying Schur's Theorem, we get:

$$\mathcal{P}_\sigma(G) \sim \frac{\left(\frac{2\sigma-2+2|G|}{A|G|/\exp(G)}\right)^r}{r!\left[\frac{2|G|}{A|G|/\exp(G)}\right]\prod_{i=1}^{r}\left[\frac{|G|(1-1/n_i)}{(A|G|/\exp(G)}\right]}$$

$$\sim \frac{2^r A|G|/\exp(G)}{r!2|G|^{r+1}\prod_{i=1}^{r}(1-1/n_i)}\sigma^r$$

$$= \frac{A2^{r-1}}{|G|^r \exp(G) r! \prod_{i=1}^{r}(1-1/n_i)}\sigma^r$$

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

Applying Schur's Theorem, we get:

$$\mathcal{P}_\sigma(G) \sim \frac{\left(\frac{2\sigma-2+2|G|}{A|G|/\exp(G)}\right)^r}{r!\left[\frac{2|G|}{A|G|/\exp(G)}\right]\prod_{i=1}^{r}\left[\frac{|G|(1-1/n_i)}{(A|G|/\exp(G))}\right]}$$

$$\sim \frac{2^r A|G|/\exp(G)}{r!2|G|^{r+1}\prod_{i=1}^{r}(1-1/n_i)}\sigma^r$$

$$= \frac{A2^{r-1}}{|G|^r\exp(G)r!\prod_{i=1}^{r}(1-1/n_i)}\sigma^r$$

This looks awfully familiar!!!!

An Asymptotic Bound for $\mathcal{P}_\sigma(G)$

Applying Schur's Theorem, we get:

$$\mathcal{P}_\sigma(G) \sim \frac{\left(\frac{2\sigma-2+2|G|}{A|G|/\exp(G)}\right)^r}{r!\left[\frac{2|G|}{A|G|/\exp(G)}\right]\prod_{i=1}^r\left[\frac{|G|(1-1/n_i)}{(A|G|/\exp(G)}\right]}$$

$$\sim \frac{2^r A|G|/\exp(G)}{r!2|G|^{r+1}\prod_{i=1}^r(1-1/n_i)}\sigma^r$$

$$= \frac{A2^{r-1}}{|G|^r\exp(G)r!\prod_{i=1}^r(1-1/n_i)}\sigma^r$$

This looks awfully familiar!!!!

Asymptotic Equivalence

Theorem

For a fixed group G, for σ within the genus spectrum of G, we have

$$\lim_{\sigma \to \infty} \frac{|\mathcal{A}_\sigma(G)|}{|\mathcal{P}_\sigma(G)|} = 1.$$

Equivalently:

Asymptotic Equivalence

Theorem

For a fixed group G, for σ within the genus spectrum of G, we have

$$\lim_{\sigma \to \infty} \frac{|\mathcal{A}_\sigma(G)|}{|\mathcal{P}_\sigma(G)|} = 1.$$

Equivalently:

- In the long run, all possible potential signatures are actual signatures for a group action.

Asymptotic Equivalence

Theorem

For a fixed group G, for σ within the genus spectrum of G, we have

$$\lim_{\sigma \to \infty} \frac{|\mathcal{A}_\sigma(G)|}{|\mathcal{P}_\sigma(G)|} = 1.$$

Equivalently:

- In the long run, all possible potential signatures are actual signatures for a group action.
- Or, eventually, satisfaction of the RH formula is sufficient for the existence of an action!

Asymptotic Equivalence

Theorem

For a fixed group G, for σ within the genus spectrum of G, we have

$$\lim_{\sigma\to\infty} \frac{|\mathcal{A}_\sigma(G)|}{|\mathcal{P}_\sigma(G)|} = 1.$$

Equivalently:

- In the long run, all possible potential signatures are actual signatures for a group action.
- Or, eventually, satisfaction of the RH formula is sufficient for the existence of an action!

So it makes sense to consider the "easy" problem first.

Asymptotic Equivalence

Theorem

For a fixed group G, for σ within the genus spectrum of G, we have

$$\lim_{\sigma \to \infty} \frac{|\mathcal{A}_\sigma(G)|}{|\mathcal{P}_\sigma(G)|} = 1.$$

Equivalently:

- In the long run, all possible potential signatures are actual signatures for a group action.
- Or, eventually, satisfaction of the RH formula is sufficient for the existence of an action!

So it makes sense to consider the "easy" problem first.