A coarse classification of Z_2^k actions

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This is joint work with Mariela Carvacho (U. Tecnica F. Santa Maria, Chile).

Let $Z_2^k \equiv$ the elementary abelian 2-group of rank $k > 2$.

We consider Z_2^k actions on surfaces of genus $g > 1$, having branch data $(0; 2, \overset{R}{\ldots}, 2) = (0; R)$, that is:

- the quotient surface is a sphere (genus 0);
- there are $R \geq k + 1$ branch points (all branch indices $= 2$), on the quotient sphere.
- the action takes place in genus $g = 2^{k-2}(R - 4) + 1$.

Equivalently, we consider surface-kernel epimorphisms

$$\rho : \Gamma(0; R) \to Z_2^k$$

specified by sets of R elements of Z_2^k which

- generate the group;
- whose sum is the identity element.

(a.k.a. 'generating vectors.')

ρ may be pre- or post-composed with group automorphisms; the generating vectors so obtained constitute a topological equivalence class of action.

We represent Z_2^k as the vector space of dimension k over the field with two elements (0, 1), hence:

- Elements are $(0, 1)$ column vectors with k entries;
- the identity $\overline{0}$ (or $\overline{0}_k$) is the column of all zeroes;
- a generating vector is a $k \times R$ matrix M of rank k with rowsum $\overline{0}$.

Here,

- 'rowsum' $=$ the column vector obtained by summing (mod 2) across the rows;
- 'matrix' means an unordered collection of columns; multiplicities are allowed.

It is convenient to compress a generating matrix M into a dependent set $\mathcal{D}_M$ in the matroid $Z_2^k - \{\overline{0}\}$, by retaining only columns of odd multiplicity, listed once each.

- if there are no columns of odd multiplicity, $\mathcal{D}_M = \emptyset$;
- if not empty, $\mathcal{D}_M$ has at least 3 columns;
- rowsum $\mathcal{D}_M =$ rowsum $M = \overline{0}$ because (mod 2):
    - columns of even multiplicity in M do not contribute to the rowsum;
    - columns of odd multiplicity contribute as if they occurred only once.

Up to change of basis and columnwise reordering, $\mathcal{D}_M$, if not empty, is a $k \times k_0$ matrix of rank r of the form

$$\mathcal{D}_M \approx \underbrace{\begin{bmatrix} I_r & A \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{k_0} \qquad 2 \leq r \leq \min\{k, k_0 - 1\}$$

where

- $k_0 \neq 1, 2$ is the number of distinct columns of M with odd multiplicity;
- A is an $r \times (k_0 - r)$ matrix with rowsum $= \overline{1}_r$.

The two parameters

- $k_0 = \#$ columns of odd multiplicity in M;
- $k_1 = \#$ columns of even multiplicity in M;

must satisfy

- $k_0 \neq 1, 2$ (for dependence of $\mathcal{D}_M$);
- $k_0 + k_1 \geq k + 1$ (for generation of Z_2^k).

A further necessary condition (if $\mathcal{D}_M \neq \emptyset$):

- existence of a $r \times (k_0 - r)$ matrix A with rowsum $\overline{1}_r$ where $2 \leq r \leq \min\{k_0 - 1, k\}$ (if $r \neq 0$).

Main Theorem (Carvacho, W.)

There is a Z_2^k action, $k > 2$, with branch data $(0; R)$ if and only if R admits an additive partition into positive integers, with k_0 odd parts and k_1 even parts, satisfying the following conditions:

- N1 $k \leq k_0 + k_1 \leq 2^k - 1$;
- N2 if $k_0 + k_1 = k$, then $k_0 = 0$;
- N3 $k_0 \neq 1, 2,\ 2^k - 3,\ 2^k - 2$.

N.B.: the actual partition (hence R, hence the genus of the action) is not specified – only the partition type

$$\{k_0 \text{ odd parts}, \quad k_1 \text{ even parts}\}.$$

Comments on the Theorem

- The number of partition types which satisfy the Theorem's conditions (for fixed k) is finite, in fact, equal to

$$2^{2k-1} - 3(2^{k-1} - 1) - \frac{k(k-1)}{2};$$

- Actions of the same partition type occur in infinitely many genera (since infinitely many R admit a partition of a given type.)
- topologically equivalent actions have the same partition; in particular the same partition type;
- hence partition types define a coarse topological equivalence with finitely many equivalence classes.

Comments on the PROOF

1. Why do partitions arise at all?

1. Each of the R branch points on the quotient sphere has 2^{k-1} preimages, each fixed by a Z_2 subgroup;
2. such a subgroup has $2g+2-4\gamma = i(2^{k-1})$ for some i;
3. the total number of fixed points is

$$R(2^{k-1}) = \sum_{i=0} m_i(2g+2-4\gamma(i)),$$

   where m_i is the number of Z_2 subgroups acting with quotient genus $\gamma(i)$;
4. cancelling 2^{k-1} yields the finite partition

$$R = \sum_{i \geq 1} m_i \cdot i,$$

   into positive parts i, where m_i is the multiplicity of the part i.

2. NECESSITY of conditions N1-N3 on the partition type is rather easy to see;
3. SUFFICIENCY :

- one has to exhibit at least one compression of the form

$$\mathcal{D}_M \approx \underbrace{\begin{bmatrix} I_r & A \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{k_0}$$

  for each admissible partition type $\{k_0, k_1\}$ $(k_0 \neq 0)$;
- the trickiest construction is A (which must have rowsum = $\overline{1}_r$ and column weights ≥ 2).

'column weight' $= \#$ of ONES in the column.

Consider again

$$\mathcal{D}_M \approx \underbrace{\begin{bmatrix} I_r & A \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{k_0} \qquad 2 \leq r \leq \min\{k, k_0 - 1\}.$$

- If $r < k$, $k_0 \leq 2^r - 1$. Moreover, $k_0 \neq 2^r - 2, 2^r - 3$ (since Z_2^r contains no dependent sets of complementary size 1, 2, resp.)
- It suffices to assume $r = k$.
- if $k_0 = 2^k - 1$, $\mathcal{D}_M = Z_2^k - \{\bar{0}\}$ and A is determined;
- $k_0 \neq 2^k - 2, 2^k - 3$ (as above).

Thus we may assume $k+1 \leq k_0 \leq 2^k - 4$, that is,

$$\mathcal{D}_M \approx \underbrace{\left[\begin{array}{c|c} I_k & \color{red}{A} \end{array} \right]}_{k+1 \,\leq\, k_0 \,\leq\, 2^k - 4}$$

(In particular, $k \geq 4$).

CLAIM: $\color{red}{A}$, with rowsum $= \overline{1}_k$, and column weights ≥ 2, can always be constructed.

- If A has up to $k-1$ columns a ‘sparse’ (= fewest ONES) matrix A can be obtained by ‘telescoping’ $k \times (k-1)$ matrices of the form(s):

$$A' \stackrel{\text{def}}{=} \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{k-1} k \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & & 1 \end{bmatrix}}_{k-1} k$$

(k even, resp., odd: here 6, resp. 7)

‘Telescoping’ = successively adding the last two columns.

- For an A with $n \geq k$ columns, let

$$n = mk + t, \quad 0 \leq t < r$$

and put

$$A = \underbrace{\left[(A')_t \quad | \quad B_1 \quad | \ldots \quad | \quad B_m\right]}_{n},$$

where

- $(A')_t = A'$ telescoped to t columns (or $= \emptyset$ if $t = 0$);
- each B_j's is a distinct 'k-block' $=$ k-set of cyclic permutations of a non-periodic column (of weight > 2).

- It suffices to consider k_0's of approximately half the maximum value:

$$k_0 \leq \frac{2^k - 2}{2} = 2^{k-1} - 1$$

(REASON: the complement of a dependent set in $Z_2^k - \overline{\{0\}}$ is also a dependent set.)

- In constructing

$$\mathcal{D}_M = \left[\begin{array}{c|c|c|c|c} I_k & (A')_t & B_1 & \ldots & B_m \end{array} \right],$$

the total rowsum of the B_j's must be $\overline{0}$ if $t \neq 0$, or $\overline{1}$ if $t = 0$.

- CLAIM: there are enough k-blocks for the construction to succeed for all $k_0 \le 2^{k-1} - 1$.
- Example $(k = 4)$: $3 \le k_0 \le 15$, e.g., $k_0 = 7$:

$$\mathcal{D}_M = [I_4 \mid A] \qquad A = A' = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- For larger k_0, take complementary sets with a change of basis:

In $Z_2^4 - \{\bar{0}\}$, let

$$C_{0,2} = \begin{bmatrix} 1 & & & 1 \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{bmatrix} \quad C_{1,1} = \begin{bmatrix} 1 & \\ & 1 \\ 1 & \\ & 1 \end{bmatrix} \quad C_{0,0,1} = \begin{bmatrix} & 1 & 1 & 1 \\ 1 & & 1 & 1 \\ 1 & 1 & & 1 \\ 1 & 1 & 1 & \end{bmatrix}$$

In general: $C_{\alpha_1,\alpha_2,\ldots,\alpha_t} \subseteq Z_2^k$ is the set of cyclic permutations of a weight t column with α_i ZEROES between the ith and $(i+1)$st ONES (counting cyclically).

Using this notation,

$$Z_2^4 - \{\bar{0}\} = \left[\quad I_4 \quad \middle| \quad C_{0,2} \quad \middle| \quad C_{1,1} \quad \middle| \, C_{0,0,1} \quad \middle| \quad \bar{1} \quad \right],$$

Continuing our example, the complement of $\underbrace{\left[I_4 \mid A' \right]}_{k_0=7}$ is

$$\left[A'' \quad \middle| C_{0,0,1} \middle| \quad \bar{1} \quad \right] \quad \text{where} \quad A'' = C_{0,2} \cup C_{1,1} - A'.$$

With the change of basis $I_4 \leftrightarrow C_{0,0,1}$ (and columnwise reordering), we obtain a dependent set for $k_0 = 15 - 7 = 8$, namely

$$\underbrace{\left[\quad I_4 \quad \middle| \quad A'' \quad \middle| \, \bar{1} \quad \right]}_{k_0=8}.$$

In this way, we obtain dependent sets of all remaining larger sizes

$$8 = 2^{k-1} \leq k_0 \leq 2^k - 4 = 12.$$

Concluding claims:

- the foregoing constructions can be systematized to produce canonical, sparse generating matrices for every partition type;
- they can be adapted to actions with positive orbit-genus, i.e., branch data $(h; R')$, $h > 0$;
- they can be extended to Z_p^k actions, for arbitrary primes p; the key is to consider dependent sets in the projective geometries

$$\mathrm{PG}_{k-1}(F_p),$$

where F_p is the field with p elements.

Thanks for your attention!