(q, n)-gonal pseudo-real Riemann surfaces

Ewa Kozlowska-Walania and Ewa Tyszkowska
University of Gdansk

3-4 October 2015

NEC groups and Fuchsian groups

$\mathcal{H}$=hyperbolic plane
$\mathcal{G}$=group of isometries of $\mathcal{H}$ including those reversing orientation
$\mathcal{G}^+$=subgroup of $\mathcal{G}$ consisting of orientation-preserving isometries

$\mathcal{G} = PGL(2, \mathbb{R})$ and $\mathcal{G}^+ = PSL(2, \mathbb{R})$

Non-euclidean crystallographic group (NEC-group) is a discrete in the topology of $\mathbb{R}^4$ subgroup of $\mathcal{G}$ with compact orbit space

An NEC-group is called a **Fuchsian group** if it is contained in $\mathcal{G}^+$ and a **proper** NEC-group otherwise.

Macbeath and **Wilkie** associated with an NEC-group Λ a **signature** of the form

$$\sigma(\Lambda) = (g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots$$

$$\ldots, (n_{k1}, \ldots, n_{ks_k})\})$$

which determines the presentation of Λ by generators:

$x_i, \ 1 \le i \le r,$	elliptic
$c_{ij}, \ 1 \le i \le k; 0 \le j \le s_i,$	reflections
$e_i, \ 1 \le i \le k,$	boundary
$a_i, b_i, \ 1 \le i \le g, \ \text{if } +,$	hyperbolic
$d_i, \ 1 \le i \le g, \ \text{if } -,$	glide reflections

and relations:

$x_i^{m_i} = 1,$	$1 \le i \le r,$
$c_{is_i} = e_i^{-1} c_{i0} e_i,$	$1 \le i \le k,$
$c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1} c_{ij})^{n_{ij}} = 1,$	$1 \le i \le k,$
	$1 \le j \le s_i,$
$x_1 \ldots x_r e_1 \ldots e_k [a_1, b_1] \ldots [a_g, b_g] = 1,$	if $+$
$x_1 \ldots x_r e_1 \ldots e_k d_i^2 \ldots d_g^2 = 1,$	if $-$.

The orbit space $\mathcal{H}/\Lambda$ is a surface of topological genus g having k boundary components and orientable or not according to the sign being $+$ or $-$.

A Fuchsian group can be regarded as an NEC-group with the signature

$$(g; +; [m_1, \ldots, m_r]; \{-\}).$$

Every NEC-group has a **fundamental region**, whose hyperbolic area $\mu(\Lambda)$ is given by

$$\begin{aligned}\mu(\Lambda) = \quad & 2\pi(\alpha g + k - 2 + \textstyle\sum_{i=1}^{r}(1 - 1/m_i) + \\ & +1/2 \textstyle\sum_{i=1}^{k} \sum_{i=1}^{s_i}(1 - 1/n_{ij})),\end{aligned}$$

where $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise.

If Γ is a subgroup of finite index in an NEC-group Λ then it is an NEC-group itself and there is the **Riemann-Hurwitz formula** which says that

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda).$$

[Preston] Every compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space $\mathcal{H}/\Gamma$ of the hyperbolic plane $\mathcal{H}$ under the action of a torsion-free Fuchsian group Γ called **surface group** with the signature

$$(g; +; [-]; \{-\}).$$

A finite group G is a group of automorphisms of $X = \mathcal{H}/\Gamma$ if and only if $G \cong \Lambda/\Gamma$ for some NEC-group Λ normalizing Γ.

[Macbeath] Let
$X = \mathcal{H}/\Gamma$,
$G = \Lambda/\Gamma = \mathrm{Aut}^{+}(X)$,
$x_1, \ldots, x_r \in \Lambda$-elliptic generators of orders $m_1, \ldots m_r$,
$\theta : \Lambda \to G$-the canonical epimorphism.
Then the number of fixed points of $g \in G$ is equal to

$$m = |N_G(\langle g \rangle)| \sum 1/m_i,$$

where the sum is taken over those i for which g is conjugate to a power of $\theta(x_i)$.

The canonical Fuchsian subgroup

Let Λ be an NEC group with the signature

$$\sigma(\Lambda) = (h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots$$

$$\ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

The subgroup Λ^+ of Λ consisting of all orientation preserving elements is called the **canonical Fuchsian subgroup** of Λ

[Singerman] The signature of Λ^+ has proper periods:

$n_{11}, \ldots, n_{1s_1}, \ldots n_{k1}, \ldots, n_{ks_k}, m_1, m_1, \ldots, m_r, m_r$

and orbit genus $h' = \varepsilon h + k - 1$, where $\varepsilon = 2$ or 1 according to if the sign in $\sigma(\Lambda)$ is $+$ or $-$.

Maximal signatures

A signature σ is said to be **maximal**, if for every NEC group Λ' with signature σ' containing an NEC group Λ with signature σ such that the Teichmüller spaces of Λ and Λ' have the same dimensions, the equality $\Lambda = \Lambda'$ holds.

Otherwise the pair (σ, σ') is called a **normal** or **non-normal pair** according to if Λ is a normal subgroup of Λ' or not.

The complete lists of normal and non-normal pairs were given by [**Bujalance, Singermann, Estevez, Izquierdo, Conder**].

If σ is a maximal signature, then there exists a **maximal NEC group** with signature σ which is not properly contained in any other NEC group.

The full automorphism group

If $G = \Lambda/\Gamma$ acts on a Riemann surface $X = \mathcal{H}/\Gamma$ with maximal signature, then G is the full automorphism group of X. In other case, Λ is properly contained with finite index in another NEC group Λ', which normalizes Γ.

If for any epimorphism $\theta : \Lambda \to G$, there exists a group G' and group embeddings $i : \Lambda \to \Lambda'$, $j : G \to G'$ and θ extends to $\theta' : \Lambda' \to G'$ such that $\theta' \cdot i = j \cdot \theta$

$$\begin{array}{ccc} \Lambda & \xrightarrow{i} & \Lambda' \\ \downarrow \theta & & \downarrow \theta' \\ G & \xrightarrow{j} & G' \end{array}$$

then

$$G = \Lambda/\Gamma \subset \Lambda'/\Gamma = G' \subseteq \mathrm{Aut}(X)$$

and so G is not the full automorphism group.

An example

$\sigma(\Lambda) = (2; -; [k]; \{-\}) : d_1, d_2, x_1,$
$\sigma(\Lambda') = (0; +; [2,2]; \{(k)\}) : x'_1, x'_2, c'_{10}, c'_{11}, e'_1$
$G = Z_{4k} = \langle \delta \rangle$
$G' = D_{4k} = \langle \tau, a : a^2 = \tau^2 = (\tau a)^{4k} = 1 \rangle$

A smooth epimorphism $\theta : \Lambda \to G$ is given by

$$\theta(x_1) = \delta^{4p},\ \theta(d_1) = \delta^{t_1},\ \theta(d_2) = \delta^{t_2}$$

for some integers t_1, t_2 and p such that t_1, t_2 are odd, $(p,k) = 1$ and $2p + t_1 + t_2 \equiv 0\ (2k)$.

$$\begin{array}{ccc} \Lambda & \stackrel{i}{\to} & \Lambda' \\ \downarrow \theta & & \downarrow \theta' \\ \mathbb{Z}_{4k} & \stackrel{j}{\to} & D_{4k} \end{array}$$

$$\begin{array}{ll} j : & \delta \mapsto \tau a \\ i : & d_1 \mapsto e'_1 c'_{10} x'_1,\ d_2 \mapsto x'_1 c'_{10},\ x_1 \mapsto c'_{10} c'_{11} \\ \theta' : & c'_{10} \mapsto \tau, c'_{11} \mapsto \tau(\tau a)^{4p}, \\ & x'_1 \mapsto \tau(a\tau)^{t_2}, e'_1 \mapsto (\tau a)^{t_1+t_2}. \end{array}$$

$$\theta' \cdot i = j \cdot \theta.$$

(q,n)-gonal Riemann surfaces

A compact Riemann surface X of genus $g > 1$ is said to be (q,n)-**gonal** if X admits a conformal automorphism ρ of prime order n such that $X/\langle\rho\rangle$ has genus q. This automorphism is called a (q,n)-**gonal automorphism**.

If $n = 2$ then (q,n)-gonal automorphism is called a q-**hyperelliptic involution**. For $(q,n) = (0,2)$ and $(1,2)$, X is called **hyperelliptic** and **elliptic-hyperelliptic**, respectively.

Pseudo-real Riemann surfaces

A **symmetry** of a Riemann surface is an anti-holomorphic involution. A surface is **symmetric** if it admits a symmetry.

Projective complex algebraic curves correspond to compact Riemann surfaces. A surface X is symmetric if the corresponding curve $\mathcal{C}_X$ is definable over $\mathbb{R}$.

A Riemann surface is called **pseudo-real** if it admits an anticonformal automorphism but no anticonformal involution. An anticonformal automorphism of order 4 is called a **pseudo-symmetry** and in this case the surface is called **pseudo-symmetric**.

Complex algebraic curves with real moduli

There is an antiholomorphic involution

$$\iota : \mathcal{M}_g \to \mathcal{M}_g$$

of the moduli space $\mathcal{M}_g$ of complex algebraic curves of genus g which maps the class of a complex curve to its conjugate.

The fixed points of ι are called **complex algebraic curves with real moduli**. The corresponding to them Riemann surfaces admit an antiholomorphic automorphism and they are either symmetric or pseudo-real.

Etayo Gordejuela studied pseudo-real surfaces with cyclic automorphism groups.

Bujalance, Conder, Costa gave many general properties of pseudo-real surfaces, such as for example existence of an asymmetric surface in any genus $g \geq 2$, description of such surfaces in genera 2 and 3 and the sharp bound on the order of the automorphism group.

The case of hyperelliptic pseudo-real surfaces was studied by **Singerman**, whilst **Bujalance, Turbek** determined defining equations for such surfaces and the special case of hyperelliptic asymmetric pseudo-symmetric surfaces is also treated.

The minimal genus problem for pseudo-real Riemann surfaces with cyclic automorphism groups was solved by **Baginski** and **Gromadzki**.

Remarks:

(i) If $G = \Lambda/\Gamma$ is an automorphism group of a pseudo-real Riemann surface $X = \mathcal{H}/\Gamma$, then Λ has signature

$$(h+1; -[m_1, \ldots, m_r]; \{-\})$$

for $h \geq 0$ and $m_1, \ldots, m_r$ dividing $|G|$ while the canonical Fuchsiam subgroup Λ^+ of Λ has signature

$$(h; +; [m_1, m_1, \ldots m_r, m_r]; \{-\}).$$

(ii) The order of an anticonformal automorphism δ of X is divisible by 4. Any conformal automorphism in $\langle\delta\rangle$ has an even number of fixed points.

Theorem 1: The cyclic group $\mathbb{Z}_{4k} = \Lambda/\Gamma$ generated by an anticonformal automorphism δ acts on a pseudo-real Riemann surface $X = \mathcal{H}/\Gamma$ of genus g with signature

$$(h+1; -; [m_1, \ldots, m_s, 2k, \overset{l}{\ldots}, 2k]; \{-\}), \text{ where}$$

(i) $g = 2k[h + \sum_{i=1}^{s}(1 - \frac{1}{m_i})] + (l-1)(2k-1)$,
(ii) $h \equiv g$ (2),
(iii) if $h = 0$ then $l \neq 0$ or there exists a subset $\{i_1, \ldots, i_{s'}\} \subset \{1, \ldots, s\}$ such that
$(m_{i_1} \cdot \ldots \cdot m_{i_{s'}})/\text{g.c.d}(m_{i_1}, \ldots, m_{i'_s}) = 2k$.
(iv) $(h, s, l) \neq (1,1,0), (0,1,1)$ and $(0,2,0)$.

proof:

(i) the Riemann-Hurwith formula for (Λ, Γ);
(ii) the necessary condition for the existence of $\theta : \Lambda \to \mathbb{Z}_{4k}$;
(iii) $\mathbb{Z}_{2k} = \Lambda^+/\Gamma$ acts with signature

$$(h; m_1, m_1 \ldots, m_s, m_s, 2k, \overset{2l}{\ldots}, 2k)$$

and so for $h = 0$ it is generated by θ-images of elliptic generators of Λ^+;
(iv) if the signature is non-maximal, then X is symmetric(see Example 1).

Lemma 1: If a (q,n)-gonal automorphism of a Riemann surface X of genus g has an even number of fixed points on X, then there exists an integer β in the range

$$-1 \leq \beta \leq g/(n-1)$$

such that n divides $g+\beta$ and

$$q = (g+\beta)/n - \beta. \quad (1)$$

proof: The group generated by a (q,n)-gonal automorphism ρ of a Riemann surface X of genus g acts with the signature $(q; n, \overset{m}{\ldots}, n)$ for

$$m = 2 + 2(g-nq)/(n-1).$$

By Macbeath's Theorem ρ has m fixed points. If m is even then for $\beta = m/2 - 1$ we get (1).

Theorem 2: A pseudo-real Riemann surface X of genus g with automorphism group $\mathbb{Z}_{4k}$ is (q,n)-gonal for any prime $n>2$ dividing k and

$$q = 1 + \frac{g - g_{(n)}}{n} - [(n - g_{(n)}) + (n-1)(d-1)], \qquad (2)$$

where $g_{(n)} = g \mod n$ and d is an integer in the range

$$(g_{(n)} - 1)/n \leq d \leq [g + g_{(n)}(n-1)]/[n(n-1)].$$

proof: The element $\rho = \delta^{\frac{4k}{n}} \in \mathbb{Z}_{2k}$ has an even number of fixed points. So by Lemma 1, the genus q of $X/\langle\rho\rangle$ is equal to

$$q = (g+\beta)/n - \beta$$

for some integer β in the range

$$-1 \leq \beta \leq g/(n-1)$$

for which $g+\beta$ is divisible by n. Such β must be $\beta = dn - g_n$ for $g_{(n)} = g \mod n$ and some integer d. Thus we get (2).

Theorem 3: For given $g \geq 2$ and prime $n > 2$, the upper sharp bound for an integer q such that there exists a (q, n)-gonal pseudo-real Riemann surface of genus g is equal to

$$q_{\text{max}} = \begin{cases} g_{(n)} + \frac{g - g_{(n)}}{n}, \ \textit{if} \ g_{(n)} = 0 \text{ or } 1, \\ g_{(n)} + \frac{g - g_{(n)}}{n} - (n-1), \ \textit{if} \ \ g_{(n)} \geq 2. \end{cases}$$

Conditions on g for which q_{max} is attained:

(i) If $g_n = 0$ or 1, then q_{max} is attained for any g such that

$$g - g_{(2)} \equiv 0 \ (n)$$

and the surface can be pseudo-symmetric or not.

(ii) If $g_n \geq 2$, then q_{max} is attained in pseudo-symmetric case for any $g > (n+1)(n-2)$, and in not pseudo-symmetric case for any g such that

$$g \geq (n-2)(4n-1) \text{ and } g + n - g_{(n)} \equiv 0 \ (4).$$

Theorem 4: If the group $G = \mathbb{Z}_{4k} = \langle \delta \rangle$ acts on a Riemann surface X with signature

$$(h+1; -; [m_1, \ldots, m_s, 2k, \overset{l}{\ldots}, 2k]; \{-\}),$$

then X is (q, n)-gonal for any prime n dividing $4k$ and

$$q = 1 + \frac{2k}{n}(h+s+l-1) - (\sum_{i=1}^{s} \frac{2k}{\beta_i m_i} + l), \quad (3)$$

where $\beta_i = 1$ or $\beta_i = n$ depending on whether n divides m_i or not. In particular, X is p-hyperelliptic for

$$p = 1 + k(h+s+l-1) - (\sum_{i=1}^{s} \frac{\gamma_i k}{m_i} + l),$$

where $\gamma_i = 2$ or 1 according to if m_i is even or odd.

proof: $\rho = \delta^{4k/n}$ has $m = 2 + (2g - 2nq)/(n-1)$ fixed points, where q is the genus of $X/\langle \rho \rangle$. Compering number m with Macbeath's formula and using the condition (i) of Theorem 1, we get (3).

Theorem 5: Let $c, r \geq 0$ and $k \geq 2$ be integers such that

$$g = c(2k-1) + r \geq 2$$

and $(c, r, k) \neq (0, k, 1), (0, k+1, 1)$ when k is odd. Let $r_{(k)} = r \mod k$.

Then for any integer γ in the range

$$0 \leq \gamma \leq (r - r_{(k)})/k, \text{ such that}$$

$$\beta = c - r + \gamma k \geq -1$$

there exists a complex algebraic curve $\mathcal{C}$ of genus g with real moduli such that the corresponding Riemann surface $X_{\mathcal{C}}$ is (q, n)-gonal for any prime $n > 2$ dividing k and

$$q = \frac{g + \beta}{n} - \beta.$$

The surface $X_{\mathcal{C}}$ is symmetric if $(c, r, \gamma) = (1, 0, 0)$ or $(0, k+1, 1)$, $(0, k, 1)$ for even k, and is pseudo-real otherwise.

Corollary: For any integers

$$k \geq 3 \text{ and } g \geq (k-2)(2k-1),$$

there exists a pseudo-real Riemann surface of genus g with automorphism group $\mathbb{Z}_{4k}$.

The surface is (q,n)-gonal for any prime $n > 2$ dividing k and

$$q = \frac{g+\beta}{n} - \beta,$$

where $\beta = (g-r)/(2k-1) - r_{(k)}$ for

$$r = g \mod (2k-1) \text{ and } r' = r \mod k.$$

Theorem 6: For given $g \geq 4$, let $g_{(2)} = g$ mod 2, and let ε be an integer in the range

$$0 \leq \varepsilon \leq (g + 3g_{(2)})/6.$$

Then for any c in the range

$$2\varepsilon - g_{(2)} \leq c \leq (g - g_{(2)} + 2\varepsilon)/4$$

such that c divides $\frac{g-g_{(2)}}{2} + \varepsilon$ and

$$c \neq 1 \text{ if } (g_{(2)}, \varepsilon) = (1, 1),$$

there exists a pseudo-real Riemann surface of genus g with automorphism group $\mathbb{Z}_{4k}$ for

$$k = (g - g_{(2)} + 2\varepsilon)/2c.$$

The surface X is (q, n)-gonal for any prime $n > 2$ dividing k and

$$q = (g - g_{(2)} + 2\varepsilon)/n + g_{(2)} - 2\varepsilon.$$

Corollary : Let $g_{(2)} = g \mod 2$ for $g \geq 4$. Then for any prime $n > 2$ dividing $(g - g_{(2)})/2$ and any integer α in the range

$$0 \leq \alpha \leq (g - g_{(2)})/4n$$

there exists a pseudo-real $(q, n) -$ gonal and $p -$ hyperelliptic Riemann surface of genus g, where

$$q = \frac{g - g_{(2)}}{n} + g_{(2)}$$

and

$$p = (2n\alpha + g_{(2)}).$$

Theorem 7: A pseudo-real Riemann surface of genus g is p-hyperelliptic for some integer p having the same parity as g in the range

$$g_{(2)} \leq p \leq p_{\text{max}},$$

where $g_{(2)} = g \mod 2$ and

$$p_{\text{max}} = \begin{cases} (g + g_{(2)})/2 & if \;\; g - g_{(2)} \equiv 0\ (4), \\ (g + g_{(2)})/2 - 1 & if \;\; g - g_{(2)} \equiv 2\ (4) \end{cases}$$

Conversely, for any such p and $g \geq 12$, there exists a p-hyperelliptic pseudo-real Riemann surface X of genus g, and it can be pseudo-symmetric or not.

Pseudo-real Riemann surfaces with automorphism group $\mathbb{Z}_{4n}$ for prime $n > 2$.

Theorem 8: A pseudo-real Riemann surface X of genus g has an aconformal automorphism δ of order $4n$ iff $g \notin \{2n-1, n-1, 2n-2\}$ and there is an integer q in the range

$$0 \leq q \leq (g-1)/n + 1 \text{ such that}$$

(i) $q \equiv g \ (n-1)$,
(ii) $q \neq 2$ if $g = n+1$,
(iii) $q \neq 1$ if $(g-n)/(n-1)$ is even.

The element $\rho = \delta^4$ is a (q, n)-gonal automorphism of X with $2\beta + 2$ fixed points, where $\beta = (g - nq)/(n-1)$.

Theorem 9: The group $\mathbb{Z}_{4n}$ acts on a pseudo-real Riemann surface X with signature

$$(h, 1+\alpha; n, \overset{\frac{\beta-\alpha}{2}}{\ldots}, n, 2, \overset{q-(\alpha+2h)}{\ldots}, 2),$$

where q satisfies the conditions of Theorem 8,
$\beta = (g - nq)/(n-1)$,
$h \equiv q\ (2)$, $0 \leq h \leq (q+1)/2$
$\alpha \equiv \beta\ (2)$ and $-1 \leq \alpha \leq \min(\beta, q - 2h)$.
The surface X is (q, n)-gonal and p-hyperelliptic for $p = nh + (\beta + \alpha)(n-1)/2$.

Table 1: The values of p for $q \in \{0, 1, 2\}$.

q	h	α	p	Conditions
0	0	0	$\beta(n-1)/2$	$\beta \geq 0$ even
	0	-1	$(\beta-1)(n-1)/2$	$\beta \geq 1$ odd
1	1	-1	$(\beta+1)(n-1)/2$	$\beta \geq 1$ odd
2	0	0	$\beta(n-1)/2$	$\beta \geq 0$ even
	0	-1	$(\beta-1)(n-1)/2$	$\beta \geq 1$ odd
	0	1	$(\beta+1)(n-1)/2$	$\beta \geq 1$ odd
	0	2	$(\beta+2)(n-1)/2$	$\beta \geq 0$ even

Table 2: The values of p for $\beta \in \{-1, 0, 1, 2\}$.

β	α	p	Conditions
-1	-1	$n(h-1)n+1$	$1 \leq h \leq (q+1)/2,$
0	0	nh	$0 \leq h \leq q/2$
1	-1	nh	$0 \leq h \leq (q+1)/2$
1	1	$n(h+1)-1$	$0 \leq h \leq (q-1)/2$
2	0	$n(h+1)-1$	$0 \leq \ h \leq q/2$
2	2	$nh+2(n-1)$	$0 \leq h \leq (q-2)/2$

Theorem 10: Let $g = qn+\beta(n-1)$ for $\beta, q \geq 3$, and let $q_{(4)} = q \mod 4$ and $\beta_{(2)} = \beta \mod 2$. Then an asymmetric Riemann surface X of genus g with automorphism group $G = \mathbb{Z}_{4n}$ is p-hyperelliptic, and the sharp bounds on p are equal to

$$p_{\min} = \begin{cases} \frac{g+n(2-q)}{2}, & \beta \text{ even}, q \text{ odd}, \\ \frac{g+1+n(1-q)}{2}, & \beta \text{ odd}, q \text{ odd }, \\ \frac{g-nq}{2}, & \beta \text{ even}, q \text{ even }, \\ \frac{g+1-n(1+q)}{2}, & \beta \text{ odd}, q \text{ even }, \end{cases}$$

and

$$p_{\max} = \begin{cases} \frac{g-\beta_{(2)}(n-1)-q_{(4)}}{2}, & q_{(4)} = 0, 2, \\ \frac{g+1}{2}, & q_{(4)} = 1, \beta \text{ odd}, \\ \frac{g-\beta_{(2)}(n-1)-4+q_{(4)}}{2}, & \text{other cases.} \end{cases}$$

Theorem 11: For given integers $q, \beta \geq 3$, there exists a pseudo-real p-hyperelliptic Riemann surface of genus $g = qn + \beta(n-1)$ for

$$p = p_{\mathsf{max}} - 2r - s(n-1),$$

where r and s are nonnegative integers satisfying the following conditions:

Table 3: Conditions on r and s

$q_{(4)} = 0$	$r \leq \frac{q}{4}$	$2r - \frac{\beta+\varepsilon}{2} \leq s \leq 2r$
$q_{(4)} = 2$	$r \leq \frac{q-2}{4}$	$2r + 1 - \frac{\beta+\varepsilon}{2} \leq s \leq 2r + 1$
$q_{(4)} = 3$	$r \leq \frac{q-3}{4}$	$2r - \frac{\beta-\varepsilon}{2} \leq s \leq 2r$
$q_{(4)} = 1, \beta$ even	$r \leq \frac{q-5}{4}$	$2r + 1 - \frac{\beta}{2} \leq s \leq 2r + 1$
$q_{(4)} = 1, \beta$ odd	$r \leq \frac{q-1}{4}$	$2r - 1 - \frac{\beta+1}{2} \leq s \leq 2r - 1$

The action of $\mathbb{Z}_8$ on pseudo-real Riemann surfaces

Theorem 12: The cyclic group $G = \mathbb{Z}_8$ acts on an asymmetric Riemann surface X of genus $g \geq 2$ with signature

$$(h+1; -; [2, \overset{(g-1-3l)/2-4h}{\ldots}, 2, 4, \overset{l}{\ldots}, 4]; \{-\})$$

for some pair $(l, h) \neq (0, 0)$ of nonnegative integers such that $(l, h) \neq (0, 1)$ for $g = 11$,
$(l, h) \neq (1, 0)$ for $g = 6$,
l has parity different from g,
h has the same parity as g,
$g - 1 - 3l \geq 8h$.

The surface X is p-hyperelliptic for

$$p = 4h + 1 + l.$$

Corollary: There are five genera less than 13 of pseudo-real Riemann surfaces with full automorphism group Z_8, namely $4, 8, 9, 10, 12$, and the degrees of hyperellipticity of such surfaces are listed below:

p	g
2	$4, 8, 10, 12$
4	$10, 12$
5	9
6	12

Theorem 13: For given $g \geq 13$, let

$$\varepsilon = g \mod 2 \text{ and } \gamma = g \mod 6.$$

Let $l_{\max}$ and s_k be integers given in Table below for k in the range $0 \leq k \leq l_{\max}/2$. Then for every such k and s in the range $0 \leq s \leq s_k$, there exists an asymmetric p-hyperelliptic Riemann surface of genus g with full automorphism group Z_8, where

$$p = 4\varepsilon + 8s + l_{\max} - 2k + 1.$$

β	$l_{\max}$	s_k
$\beta = 1$	$(g-1)/3 - 4$,	$(2+3k)/8$
$\beta = 4$	$(g-1)/3$,	$3k/8$
$\beta \not\equiv 1 \mod 3$	$(g-\beta)/3 - (1+\varepsilon)$,	$(6l + 2 + \beta - 5\varepsilon)/16$.