The number of real ovals of a cyclic cover of the sphere

joint work with Javier Cirre

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$$y^n - f(x) = 0$$

defines a Riemann surface. How many ovals are fixed by complex conjugation?

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References:

Costa and Izquierdo: Determined the topological features of all possible symmetries of p-cyclic covers of the Riemann sphere for p prime.

Izquierdo and Shaska have recently studied covers of this type.

$$F(x, y) = y^n - f(x) = 0$$

If $F(a, b) = 0$, and either $\frac{\partial F}{\partial x}(a, b) \neq 0$ or $\frac{\partial F}{\partial y}(a, b) \neq 0$, then

1. (a, b) is a nonsingular point.

2. there is one point on the Riemann surface corresponding to (a, b)

3. either $x - a$ or $y - b$ is a local parameter.

If $F(a,b) = \frac{\partial F}{\partial x}(a,b) = \frac{\partial F}{\partial y}(a,b) = 0$, then (a,b) is a singular point and there may be one or more points lying over it. Note that $dF/dy = ny^{n-1}$, so the singular points lie among the roots of $f(x)$.

So assume

$$f(x) = cg(x)(x - a_1)^{d_1 m_1} \ldots (x - a_k)^{d_k m_k}$$

where $c = \pm 1$, $g(x)$ is a product of terms of the form $(x - \alpha)(x - \overline{\alpha})$ with non-real α, each a_i is real and $\gcd(n, d_i m_i) = d_i$.

$$y^n - f(x) = 0,$$
$$f(x) = cg(x)(x - a_1)^{d_1 m_1} \ldots (x - a_k)^{d_k m_k}$$

If n is odd, there is one fixed oval. Assume n is even.

For each real r with $f(r) > 0$, there are s^+ and s^-, positive and negative respectively, such that $(x, y) = (r, s^{\pm})$ are solutions to $y^n - f(x) = 0$. How do the branches (r, s^+) and (r, s^-) join over the points $(a_1, 0), \ldots, (a_k, 0)$?

Fix i. Let $y^n-(x-a)^{dm}h(x)=0$, where $d=\gcd(n,dm)$ and $h(x)$ is a polynomial that is relatively prime to $x-a$. Note that $y^n/(x-a)^{dm}-h(x)=0$ and that $h(a)\neq 0$. Define

$$w := \frac{y^{n/d}}{(x-a)^m}, \text{ so } w^d - h(x) = 0.$$

Let P be any point of X that lies over $(a,0)$, then $w^d(P)=h(a)\neq 0$, so w has d values when $x=a$, namely $\zeta e^{2\pi ik/d}$ where $\zeta>0$, $\zeta^d=h(a)$ and $k=0,1,\ldots,d-1$. Therefore there must be d different points of X at which w takes on these values.

Since

$$w = \frac{y^{n/d}}{(x-a)^m}, \text{ and } w(P) \neq 0,$$

$\frac{n}{d}\text{ord}_P(y) = m\text{ord}_P(x-a)$, combined with $\text{gcd}(\frac{n}{d}, m) = 1$, yield there are exactly d points lying over $(a, 0)$, that $x - a$ has order n/d at each of them, and y has order m at each of them.

Local parameter at P: First note that if $m = 1$, then y is a local parameter. For the general case, let u and v be integers such that $mu + vn/d = 1$. Then $t := y^u(x-a)^v$ has order $mu+vn/d = 1$.

Change of coordinates:

$$w = \frac{y^{\frac{n}{d}}}{(x-a)^m}, t = y^u(x-a)^v$$

$$x - a = \frac{t^{\frac{n}{d}}}{w^u}, y = t^m w^v$$

This is a real change of coordinates. If P lies over $(a, 0)$, recall $w(P) = \zeta e^{2\pi ik/d}$. If d is **odd**, there exists a unique real point P over $(a, 0)$, $f(x) = (x-a)^{dm}h(x)$ changes sign at $x = a$; if $f(x) > 0$, the two branches of solutions (r, s^+) and (r, s^-) of $y^n - f(x) = 0$ meet at the real point lying over $x = a$. There are no fixed points in a neighborhood of the other side of $x = a$.

The interesting case is when d is **even**. Then there are two real points Q^+ and Q^-, where w is positive and negative, respectively, lying over $(a, 0)$. We now prove that the behavior of the fixed ovals depends entirely on the parity of m, and n/d and the signs of y and $x-a$. Define sign(c) to be $+$ if $c > 0$ and $-$ if $c < 0$. Near $(a, 0)$,

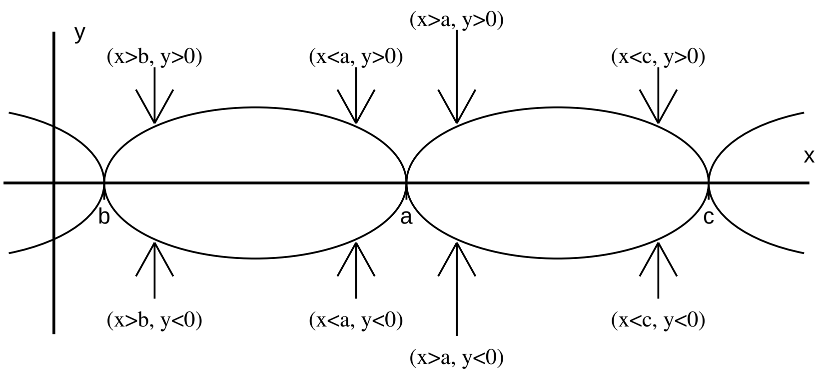
$$(t, w) = \left((x-a)^v y^u, \frac{y^{\frac{n}{d}}}{(x-a)^m} \right).$$

Therefore, t and w have the signs

$$(\text{sign}(t), \text{sign}(w)) = \left(\text{sign}(x-a)^v \text{sign}(y)^u, \frac{\text{sign}(y)^{\frac{n}{d}}}{\text{sign}(x-a)^m}\right).$$

Define $\lambda = \text{sign}(x-a)^m \text{sign}(y)^{n/d}$ and note that the fixed oval goes through Q^λ. When an oval reaches Q^λ, t changes sign, but in a neighborhood, the sign of w remains the same.

The picture in the x, y plane is this.

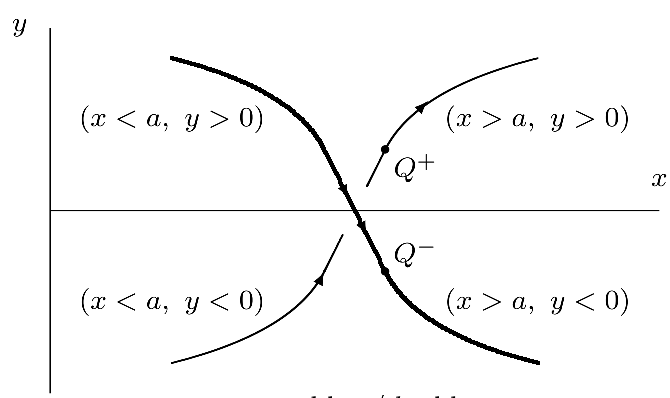


We translate this into the t, w plane. Consider $(a, 0)$ and assume d is even, so Q^+ and Q^- lie over $(a, 0)$.

Recall that $mu + v\frac{n}{d} = 1$. We have three cases. 1. m is odd and n/d is odd. 2. m is even and n/d is odd. 3. m is odd and n/d is even.

$$\lambda = \text{sign}(x-a)^m \text{sign}(y)^{n/d}$$

m is odd and n/d is odd: The branches $x > a$, $y > 0$, and $x < a$, $y < 0$ go through Q^+. The branches $x > a$, $y < 0$ and $x < a$, $y > 0$ go through Q^-.



m odd, n/d odd

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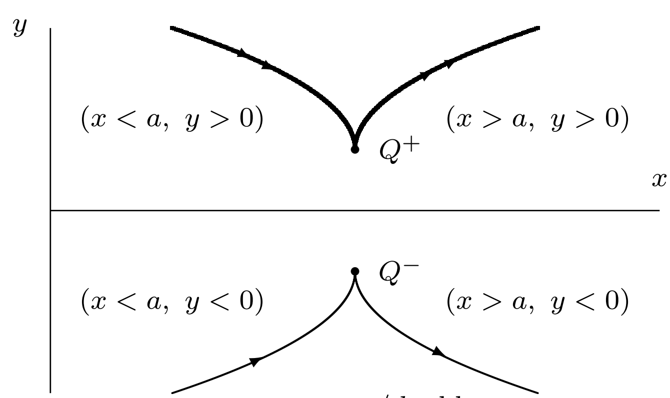

m even, n/d odd

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m is odd and n/d is even: The branches $x > a$, $y > 0$ and $x > a$, $y < 0$ go through Q^+. The branches $x < a$, $y > 0$ and $x < a$, $y < 0$ go through Q^-.

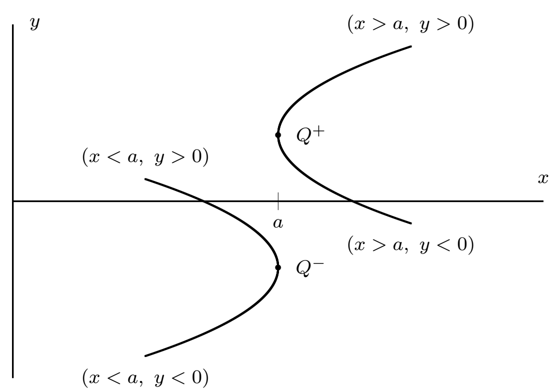

Figure 1: d and n/d even: a *bi-oval root* of f.

We say that a is a *bi-oval* root of f.

A similar result holds when $x = \infty$ where md, instead of referring to the degree of $(x - a)$, is the degree of $f(x)$.

Summary: If d is odd, $(a, 0)$ marks the end of one oval or the beginning of one oval, but not both.

If d is even and n/d is even, the $(a, 0)$ is a bi-oval root of f; it marks the end of one oval and the beginning of another oval.

If d is even and n/d is odd, then $(a, 0)$ does not mark the beginning or end of an oval.

Theorem: Let $y^n = f(x)$ be a defining equation of an n-cyclic cover of the Riemann sphere with n an even divisor of deg(f) and f monic with real coefficients. Let us assume that f does not change sign. Let b be the number of bi-oval roots of f. Then the number $\|\sigma\|$ of ovals fixed by complex conjugation σ is

$$\|\sigma\| = \begin{cases} b & \text{if } b > 0; \\ 1 & \text{if } b = 0 \text{ and } \deg(f)/n \text{ is odd;} \\ 2 & \text{if } b = 0 \text{ and } \deg(f)/n \text{ is even.} \end{cases}$$

Theorem: Let $y^n = f(x)$ be a defining equation where n even, $n|\deg(f)$ and f is monic. Let $a_1 < a_2 < \cdots < a_{2s}$ be the real roots of f with odd multiplicity, ($s > 0$) and let b_j be the number of bi-oval roots of f which lie in the interval (a_{2j}, a_{2j+1}) for $j = 1, \ldots, s-1$. Let also b_0 and b_s be the number of bi-oval roots of f which lie in the intervals $(-\infty, a_1)$ and (a_{2s}, ∞), respectively. Then the number $\|\sigma\|$ of ovals fixed by complex conjugation σ is

$$\|\sigma\| = s + \sum_{j=0}^{s} b_j.$$