Kulkarni's Theorem and finite groups acting on a surface of genus g with $g - 1$ prime.

Thomas Tucker
Colgate University, Hamilton, New York, 13346

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Corollary to Kulkarni's Theorem G acts on almost all surfaces if and only if G is almost Sylow cyclic and does not contain $C_2 \times C_4$.

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p-groups acting on surface of genus $p+1$

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For second, we have $6 \geq 3^n(-2/3)$ so $n = 2$ and this works with $C_3 \times C_3$.