Jacobi type formulas for cyclic covers of $\mathbb{CP}^1$

Preliminaries

My goal is to investigate generalizations of the following relations for elliptic curves: $K = \frac{\pi}{2}\theta_3(\tau)$. θ is one of the theta functions. K can be considered as the a period of the non-vanishing holomorphic differential $\frac{dx}{y}$. (Recall that elliptic curves are given in the form : $y^2 = x(x-1)(x-\lambda)$. so $\frac{dx}{y}$ is non-vanishing) In general if $a_1 ... a_g, b_1 ... b_g$ is the normalized homology and $v_1 ... v_g$ are certain non-normalized set of holomorphic differentials can we express $\int_{a_i} v_j$ through theta functions and normalized periods?

Motivation

There is a method to this madness: We like to find effective expressions to these as these formulas should give us better grasps on the connection between KDV of integrable PDEs and theta functions. As we will see below they should also lead to generalizations of the following identity:

$$\theta_1'(\tau) = \theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \tag{1}$$

That was a topic of continuous fascination since it was discovered by Jacobi 200 years ago.

How do we get such formulas

Rather than to go through tedious explanations and definitions of theta function properties I like to explain how such formulas can be obtained. The key to think about theta functions as uniformzers of functions on Riemann surface. That is just as any function of CP^1 can be expressed through $\frac{\prod(z-a_i)}{\prod(z-b_i)}$ any function on a general Riemann surface $\mathbb{X}$ can be written as a rational product of theta functions.
z lives on CP^1 itself but θ lives on a g dimensional variety associated with $\mathbb{X}$ It was Riemann's greatest contribution to invent such functions and to provide a parametrization of any other function on the surface through a quotient of θ and its translates.

Main idea

The main idea of getting relations between θ and its derivatives is very simple. Assume for a moment that you know that $\frac{f(x)}{g(x)} = p(x)$ Now for a point x_0 further assume that $g(x_0) \neq 0$ but $g'(x_0) = 0$. Taking the quotient derivative on both sides we get that:

$$\frac{f'(x_0)}{g(x_0)} = p'(x_0) \tag{2}$$

Or:

$$f'(x_0) = p'(x_0)g(x_0) \tag{3}$$

And we are done provided we have a nice expression for $p(x)$.

Riemann Surface Case

Of course for Algebraic curves (Riemann surfaces) life isn't that simple. Because the LHS is a transcendental crazy expressions in theta functions while the LHS is an algebraic expression on a curve. But it becomes just a book keeping exercise using a repetitive definition of the chain rule to connect all the quantities involved.

N=2 case

This mechanism works nicely in the most straightforward generalization of elliptic curves namely the hyper-elliptic case (These have some application in cryptography though I am not an expert) In this case we have that : $y^2 = \prod_{i=1}^{2g+1}(x - \lambda_i)$. This is a curve of genus g (one of the branch points is ∞) The basis for holomorphic differentials are: $du_i(P) = \frac{x^i dx}{y}$ and $i = 1...g$. For any homology basis define $\rho_{ij} = \int_{a_i} du_j(P)$ Then $v_i = \rho^{-1} du_i(P)$ are the holomorphic normalized differentials. (ρ is the matrix of a periods of du_i.)

Expression for the Jacobi Inverse Problem

If $w_1 ... w_g$ are components of a vector in $\mathbb{C}^g$ we can write the Jacobi inverse problem as: $\sum_{i=1}^{g} \int_{\infty}^{P_i} v_j = w_j$ Assuming that the solution for the Jacobi problem are points x_i, y_i we have that: $\frac{\partial x_i}{\partial w_j} = \rho V^{-1}$ and V^{-1} is the inverse matrix to the matrix defined by: $s_{ij} = \frac{x_j^i}{y_j}$.

Hyperelliptic curve - Thomae second formula - Derivation Outline

For hyper-elliptic curves if we form a partition of $2g+1$ such one part contains g points and the other $g+1$ Then we know that $\sum_{i=1}^{g} e_i - K_\infty$ (K_∞ is the vector of Riemann constants)is a non vanishing even characteristics whose first derivative vanishes! This is exactly the situation I described above. Note that $y^2 = \prod_{i=1}^{2g+1}(x-\lambda_i)$ is a double cover of CP^1 and hence any function that has double poles and zeros at the branch points collapses to a function on the Sphere. Writing such functions as quotients of theta and carrying the procedure we outlined above results in the following theorem:

Thomae formula

Let $v_1, \ldots, v_g$ be the normalized holomorphic integrals. Let $\mathcal{I}_1 = \{e_1, \ldots e_{g-1}\}$ be the set of any $g-1$ branch points and K_{P_0} be the vector of Riemann constants which base point is branch point $P_0 \ni \mathcal{I}$ then,

$$\varepsilon(\mathcal{I}_1) = \sum_{j=1}^{g-1} \int_{P_0}^{e_j} \mathrm{d}v_j - K_{P_0}$$

. is non-singular odd half-period which characteristic we denote as $[\varepsilon]$

$$\frac{\partial}{\partial v_j}\theta[\varepsilon(\mathcal{I}_1)] = \epsilon\sqrt{\frac{\det\mathcal{A}}{2^{g+2}\pi^g}}\Delta(\mathcal{I}_1)^{1/4}\Delta(\mathcal{J}_1)^{1/4}\sum_{i=1}^{g}\mathcal{A}_{j,i}s_{g-j}(\mathcal{I}_1), \tag{4}$$

$$j = 1, \ldots, g, \tag{5}$$

Thomae Formula - Continued

where $\mathcal{A}$ be the $g \times g$ matrix of a-periods with entries $\mathcal{A}_{i,j}$, $\Delta(\mathcal{I}_1)$, $\Delta(\mathcal{J}_1)$ are Vandermonde determinants and built on branch points indexed from sets $\mathcal{I}_1$, $\mathcal{J}_1$, $s_{g-j}(\mathcal{I}_1)$ are elementary symmetric functions of degree $g-j$, finally ϵ is 8-th root of unity.

What happens with other cyclic cover

For other cyclic covers the non-vanishing divisors on the branch points appear more than once in the non-vanishing divisors except of one case that I am aware of. So the mapping isn't clean and I it's not clear how to generalize this process when the point on a Riemann surface appears more than once. For the case $y^3 = \prod_{i=1}^{s}(x - p_i) \prod_{j=1}^{s-1}(x - q_i)$ I carried out the process as non-vanishing divisors are of a particular simple form. We are verifying the formula numerically currently

What's next

The interesting thing is the general case and for that we first need Thomae type formulas. For cyclic covers (Abelian as well) the groups that act on the space of invariant divisors and hence on the sections of the corresponding line bundles can be decomposed into one-dimensional eigenspaces. This is crucial to find divisors for which Thomae formula is obtained. But in general the irreducible representations we have aren't one dimensional so I not sure how to approach the problem. Perhaps somebody in the audience will be able to meet the challenge?