On the Hyperelliptic Branch Locus of Klein Surfaces with One Boundary Component

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Joint work with Antonio. F. Costa & Ana M. Porto

$\{\text{Riemann Surfaces}\} \longleftrightarrow \{\text{Complex Curves}\}$

$\{\text{Klein Surfaces}\} \longleftrightarrow \{\text{Real Curves}\}$

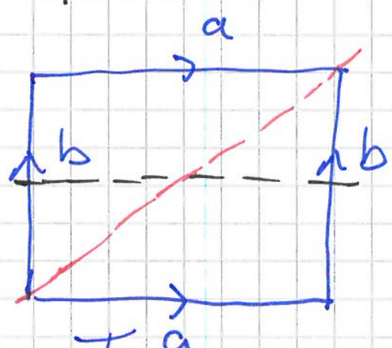
Y Klein surface $\quad Y = X/\langle\sigma\rangle \quad$ X Riemann surface

σ : antiholomorphic involution

(X, σ) = Symmetric (real) Riemann Surface

$\mathrm{Fix}(\sigma) = \dot{\bigcup}\{\text{ovals}\} \longleftrightarrow$ Boundary Comp of Y $\longleftrightarrow$ Ovals of the real curve.

Cylinder



Torus

Mobius Band

We will consider compact surfaces

$$\{\text{Riemann Surfaces}\} \xleftrightarrow[g \geq 2]{} \{\text{Fuchsian Groups}\}$$

$$\{\text{Klein Surfaces}\} \xleftrightarrow[p \geq 2]{} \{\text{NEC Groups}\}$$

- $Y = X/\langle\sigma\rangle$ X: Complex Double of Y

Y has genus g and k boundary components

X has genus $p = \alpha g - 1 + k$ $\left(\begin{array}{l} \alpha = 1, \ Y \text{ non-orientable} \\ \alpha = 2 \ \ Y \text{ orientable} \end{array} \right)$

(p: algebraic genus of Y)

Deformation $\longrightarrow$ Moduli Spaces

$\mathcal{M}_g$ = space of complex structures on orientable compact surfaces of genus g

$\mathcal{M}_t^k$ = space of dianalytic structures on surfaces with fixed topological type $t = (g, \pm, k)$

$\mathcal{M}_g$ is a (complex) orbifold of dim $3g-3$

$\mathcal{M}_t^k$ is a real orbifold of dim $3(\alpha g - 1 + k) - 3$

The spaces $\mathcal{M}_g$, $\mathcal{M}_t^k$ are connected

Mika Seppälä (1990) showed that $\mathcal{M}_g^{\mathbb{R}}$ is connected

$\mathcal{M}_g^{\mathbb{R}}$ = subspace of $\mathcal{M}_g$ formed by real Riemann Surfaces

$\mathcal{M}_g^{\mathbb{R}}$ has a spine (Costa - I 2001)

Branch loci, $\mathcal{B}_g$ and $\mathcal{B}_\varepsilon^k$ are the singular sets of $\mathcal{M}_g$ resp. $\mathcal{M}_\varepsilon^k$

The branch loci are formed by surfaces with non-trivial isometries

(Except for $g \neq 2$ or $p = \alpha g - 1 + k = 2$, where all the surfaces are hyperelliptic; one requires the surfaces to have other isometries then the hyperelliptic involution)

$\mathcal{B}_g$ and $\mathcal{B}_\varepsilon^k$ connected implies that a symmetric surface (with isometries) can be continuously deformed to another symmetric surface preserving the property of having an isometry along the whole path of deformation

Some results on connectedness / connectivity

$\mathcal{B}_g$ is connected only for $g = 3, 4, 13, 17$ & 59
(Costa - I 2010, Costa - I 2011, Bartolini - I 2012
Bartolini - Costa - I 2013)

$\mathcal{M}_g^{\mathbb{R}, Hyp}$ is connected (Seppälä 1990)
($\mathcal{B}_2$, all surfaces hyperelliptic has one isolated point
$y^5 = x^2 - x$

$\mathcal{M}_g^{p\text{-gonal}}$ is not connected in general (Costa - I 2011, Bartolini - Costa - I 2012)

$\mathcal{B}^k_{(g,+,0)}$ is connected (Bartolini - Costa - I - Porto 2010)

$\mathcal{B}^h_{(g,-,8)}$ is connected for genera up to 5
(Bujalance - Etayo - Martinez - Szepietowski 2014)

$\mathcal{B}^k_{(g,+,k)}$ is connected for $2g+k>3$ $(g,k)\neq(2,0)$
(Costa - I - Porto 2014)

In the proof we do not used hyperelliptic surfaces as Seppälä did 1990

One can see that $\mathcal{B}^k_{(2,-,1)}$ is not connected (from Cirre 2002)
(this surfaces are hyperelliptic)

Question is $\mathcal{B}^{k,Hyp}_{\varepsilon}$ connected ??

- $\mathcal{B}^{k,Hyp}_{(g,+,1)}$ is connected

- $\mathcal{B}^{k,Hyp}_{(g,-,1)}$ is not connected; it has $\frac{g+2}{2}$ or $\frac{g+1}{2}$ components according to the parity of g.

Klein Surfaces and NEC Groups

Y : Klein surface (may be non-orientable and with boundary) endowed with a dianalytic structure
(transition maps analytic or anti-analytic)

Y has topological type $t = (g, \pm, k)$

g = genus of the surface
$\pm$ = orientable or not
k = # of boundary components

Γ : NEC (Non-Euclidean Crystallographic) Group
is a cocompact discrete subgroup of $\mathrm{Aut}^{\pm}(\mathbb{H})$

Fuchsian group = NEC group consisting only of orientation-preserving transf.

Given an NEC group Γ $\quad \Gamma^{+} = \Gamma \cap \mathrm{Aut}^{+}(\mathbb{H})$ is Fuchsian
(Canonical Fuchsian Subgroup)

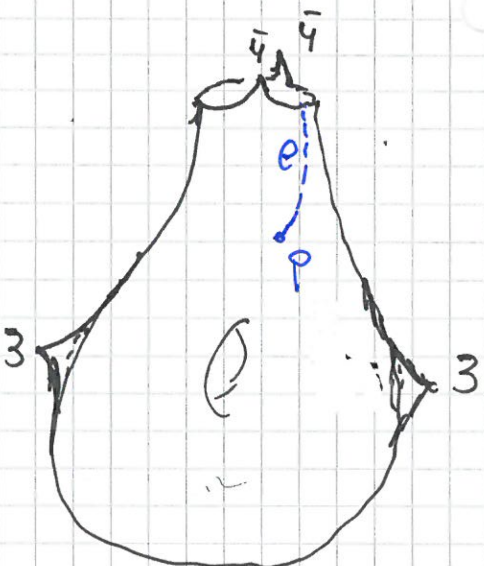
A Klein surface $Y = \mathbb{H}/\Gamma$, Γ NEC group
(If Γ proper NEC gr ; $X = \mathbb{H}/\Gamma^+$ complex double of Y)

Algebraic / Geometric Structure of Γ / $\mathbb{H}/\Gamma$ is given by the signature

$$s(\Gamma) = (g; \pm; [m_1, \dots m_r]; \{(n_{11}, \dots n_{1s_1}) \dots (n_{k1}, \dots n_{ks_k})\})$$

g = genus of the quotient
$\pm$ = orientability quotient
m_i = order of isotropy of cone points
$(n_{j1} \dots n_{js_j})$ - boundary comp
n_{jh} = order of isotropy of corner point

+ hyperbolic generators $a_1, b_1, \dots, a_g, b_g$
* glide-reflections $d_1, \dots d_g$
* elliptic generators x_i ; $x_i^{m_i} = Id$
* connecting generators $e_1, \dots e_k$
* generating reflections :
$c_{10} \dots c_{1s_1}, \dots c_{k0}, \dots c_{ks_k}$
$e_j^{-1} c_{j0} e_j = c_{js_j}$ $\quad 1 \leq j \leq k$
* $(c_{jh-1} c_{jh})^{n_{jh}} = Id$ $\quad 1 \leq h \leq s_j$
* long relation
+ $\prod x_i \prod e_j \prod [a_h, b_h] = Id$
− $\prod x_i \prod e_j \prod d_h^2 = Id$



$$s(\Gamma) = (1, +, [3, 3], \{(4, 4)\})$$

$$\Gamma = \left\langle a, b, c_0, c_1, c_2, e, x_1, x_2 \;/\; \begin{array}{l} x_1^3 = x_2^3 = c_1^2 = c_2^2 = c_0^2 = Id \\ (c_0c_1)^4 = (c_1c_2)^4 \\ e^{-1}c_0e = c_2, \quad x_1x_2e[a,b] = Id \end{array} \right\rangle$$

If Γ has no elliptic elements, Γ surface group

$s(\Gamma) = (g; \pm; [-]; \{(-) \cdots (-)\})$ Γ NEC

$s(\Gamma) = (g; -)$ Γ Fuchsian

Given a Klein surface Y, it is isomorphic to $\tilde{Y} = \mathbb{H}/\Gamma$ with Γ a surface group

A Klein surface Y is isometric to the fundamental region $P = \mathbb{H}/\Gamma$ of the uniformizing NEC group Γ. The area of Y (of Γ) is the area of any fund. region of Γ

$$\mu(Y) = \mu(\Gamma) = \mu(P) = 2\pi\left(\alpha g - 2 + k + \sum\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum\left(1 - \frac{1}{n_{ij}}\right)\right)$$

$$s(\Gamma) = (g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}) \ldots (n_{k1} \ldots n_{ks_k})\})$$

Let Γ be a subgroup of an NEC group Λ of finite index

We have the Riemann-Hurwitz formula

$$|\Lambda : \Gamma| = \frac{\mu(\Gamma)}{\mu(\Lambda)}$$

The inclusion $i : \Gamma \hookrightarrow \Lambda$ induces a representation

$\theta : \Lambda \longrightarrow \Sigma_{\Gamma\text{-cosets}}$ that describes (it is the monodromy)

the (orbifold)-covering $f : X = \mathbb{H}/\Gamma = P_\Gamma \longrightarrow Y = \mathbb{H}/\Lambda = P_\Lambda$

A finite group G is a group of automorphisms of a Klein surface $Y = \mathbb{H}/\Gamma$, Γ a surface group, if there are Λ NEC gr.

$\theta : \Lambda \longrightarrow G$

θ monodromy of the Galois covering $\mathbb{H}/\Gamma \longrightarrow \mathbb{H}/\Lambda$

(Bujalance, Etayo, Gromadzki 83-90, structure of Λ for distinct G,

(Singerman, Hoare, 78-90, structure of Γ depending on Λ, and θ)

Hyperelliptic Klein Surfaces.

* A Riemann surface X is hyperelliptic if it admits a conformal involution φ, s.t. $X/\langle\varphi\rangle$ is Riemann Sphere

* A (bordered) Klein surface Y is hyperelliptic if it there is an involution φ of Y such that $Y/\langle\varphi\rangle$ has algebraic genus 0 (i.e it is a disc (or a projective plane))

  φ is called the hyperelliptic involution

In terms of NEC (Fuchsian) groups Y/Γ, Γ surface gr,

Γ is an index two subgroup of Λ with $s(\Lambda) =$:

Y R.S $s(\Lambda) = (0; 2 \overset{2g+2}{-\!-} 2)$

Y K.S. of topological type $t = (g, \pm k)$ i) $g = 0$ $s(\Lambda) = (0; +; [-]; \{(2 \overset{2k}{-\!-} 2)\})$

ii) $g \neq 0$, Y orientable $s(\Lambda) = (0; +; [2 \overset{2g+k}{-\!-} 2]; \{(-)\})$

iii) ~~Y~~ non-orientable $s(\Lambda) = (0; +, [2 \overset{g}{-\!-} 2]; \{(2, \overset{2k}{-\!-} 2)\})$

In case of surfaces with one boundary $(0; +; [2 \overset{2g+1}{-\!-} 2] \{(-)\})$ or $(0; +; [2 \overset{g}{-\!-} 2] \{(2, 2)\})$

Moduli Spaces

Let s be a signature of NEC groups and Δ an abstract group isomorphic to them. The Teichmüller space:

$$T(s) = \{ r : \Delta \hookrightarrow \mathrm{Aut}^{\pm}(\mathbb{H}) : r(\Delta) \text{ NEC group with signature } s \} \Big/ \sim_{\mathrm{Aut}^{\pm}(\mathbb{H})}$$

$(r_1(\Delta) \sim r_2(\Delta)$ $\exists \gamma \in \mathrm{Aut}(\mathbb{H})$ $\forall g \in \Delta$ $r_1(g) = \gamma r_2(g) \gamma^{-1})$

If $s = (g; \pm; [m_1 \dots m_r]; \{(n_{11} \dots n_{1s_1}) \dots (n_{k1} \dots n_{ks_k})\})$ $T(s)$ is homeom to a ball of dim $d(s) = 3(\alpha g - 1 + k) - 3 + 2r + \sum_1^k s_i$

The modular group $\mathrm{Mod}(\Delta) = \mathrm{Aut}(\Delta) / \mathrm{Inn}(\Delta)$

Moduli space $\mathcal{M}_s^k = T_s / \mathrm{Mod}(\Delta)$ orbifold with (orbifold) fund gr $\mathrm{Mod}(\Delta)$

$\forall$ Klein surface of topological type $t = (g, \pm, k)$, the Teichmüller and moduli spaces

T_t where $t = (g; \pm; [-]; \{(-) \overset{k}{\dots} (-)\})$ (Δ a surface group)

$$\mathcal{M}_t^k = T_t / \mathrm{Mod}(\Delta)$$

B_t^k = branch locus of the universal covering $T_t \longrightarrow \mathcal{M}_t$

If Λ is a subgroup of Δ (abstract subgroups isomorphic to NEC(q))

Given an inclusion map $i : \Lambda \hookrightarrow \Delta$ induces an embedding $T(i) : T_s \longrightarrow T_{s'}$ where s' and s are the

$[\Gamma] \hookrightarrow [\Gamma oc]$

signatures (of (groups isomorphic to) Λ and Δ resp.

The action $\theta : \Delta \longrightarrow G$ of a finite group G on a Klein surface $Y = \mathbb{H}/\Gamma$, $\Gamma = \ker\theta$ surface group (inducing $i : \Gamma \hookrightarrow \Delta$) gives ~~an~~ determined embedding of $T_{s(\Delta)}$ in T_t

$T(i) : T_{s(\Delta)} \longrightarrow T_t$ called $T_t^{G,\theta} = Im(T(i))$

the projection of $T_t^{G,\theta}$, $B_t^{G,\theta}$ consists of the Klein surfaces admitting ~~an~~ action of G determined by θ is connected in M_t^k

$$B_t^k = \bigcup_{(G,\theta)} B_t^{G,\theta} = \bigcup \overline{B}_t^{C_p,\theta}$$

Nielsen's Realisation Theorem

p prime number

(1990 Broughton Riemann S)

We are interested $t = (g, \pm, 1)$

Notice that $\bar{\mathcal{B}}_t^{C_p, \theta} \cap \bar{\mathcal{B}}_t^{C_{p'}, \theta'} \neq \emptyset$ if there is a Klein surface admitting the action θ of C_p and the action θ' of $C_{p'}$. i.e the surface admits an action $\vec{\theta}$ of a group G containing both C_p and $C_{p'}$ such that $\vec{\theta}/_{C_p} = \theta$ and $\vec{\theta}/_{C_{p'}} = \theta'$

$$\begin{tikzcd}
 & \Lambda \arrow[rr, "\vec{\theta}"] & & G \\
 & \Lambda_2 = \vec{\theta}^{-1}(C_{p'}) \arrow[u] \arrow[rr, "\theta'"] & & C_{p'} \arrow[u] \\
\Lambda_1 = \vec{\theta}^{-1}(C_p) \arrow[uur] \arrow[rr, "\theta"] & & C_p \arrow[uur] & \\
\Gamma = \ker\theta \arrow[u] \arrow[uur] \arrow[rr] & & \langle 1 \rangle \arrow[u] \arrow[uur] &
\end{tikzcd}
\qquad : Y = \mathbb{H}/\Gamma$$

In case of orientable Klein surfaces C_2 could be a group of conformal involution or a group of a <u>reflection</u>

We ~~consider~~ consider the hyperelliptic branch locus; i.e those surfaces that contain more non-trivial isometries than the hyperelliptic involution

$$\mathcal{B}^{k,\mathrm{Hyp}}_{(g,\pm,1)} = \mathrm{proj}\left(\bigcup_{\substack{s < s(\Delta) \\ i}} T(i)(T_s)\right) = \bigcup \mathrm{proj}\left(T(i)(T_s)\right)$$

where $s < s(\Delta)$ means signatures s of NEC groups that contain the groups Δ with signatures

$s(\Delta) = (0; +; [2 \overset{g}{-} 2]; \{(2,2)\})$ for non-orientable Klein surfaces

$s(\Delta) = (0; +; [2 \overset{2g+1}{-} 2]; \{(-)\})$ for orientable Klein surfaces

Surfaces $Y = \mathbb{H}/\Gamma$, Γ surface group with $\theta : \Lambda \longrightarrow G$, $G = \mathrm{Aut}(Y)$ $\ker \theta = \Gamma$, and $\varphi \in G$ with $\theta^{-1}\langle\varphi\rangle = \Delta$. s is the signature of the groups Λ and i is the monomorphism inducing the action of G.

Before we show the connectivity of $\mathcal{B}^{u, Hyp}_{(g,\pm,1)}$ it is interesting the topological classification of involutions of Klein surfaces

If $\phi : X \longrightarrow Y$ is an involution of X the action is given by $\tilde{\Delta} = \langle \Gamma, \phi \rangle$ and $s(\Delta) = (h; \pm; [2^r]; \{(\ )^e (2^{s_1}) \cdots (2^{s_k})\})$

* ϕ has r isolated fixed points
* Each empty period-cycle corresponds to an oval (could be twisted) fixed by ϕ or to a boundary curve of Y (could be setwise fixed by ϕ)
* Each non-empty period-cycle corresponds to a chain of length s_i in Y

  A chain C of length s is a set of s arcs properly embedded in Y (i.e the ends of each arc are in the boundary of Y) such that for each boundary component B of Y either

  $C \cap B = \emptyset$ or $C \cap B$ consists of two distinct points

(Bujalance - Costa - Natanzon - Singerman 1992)

$\mathcal{B}^{k,Hyp}_{(g,-,1)}$ consits of $\frac{g+2}{2}$ connected components if g even

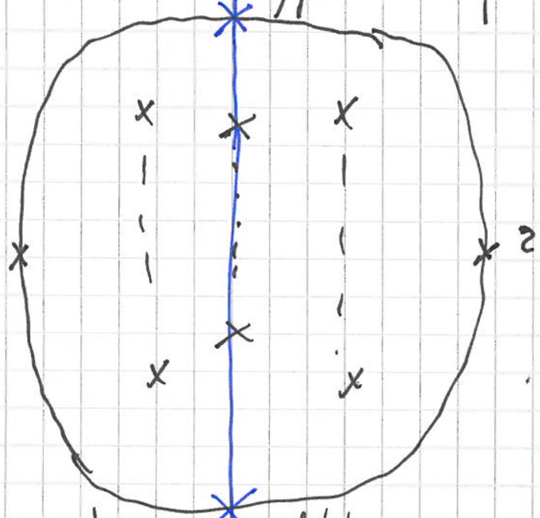
$\frac{g+1}{2}$ connected components if g odd

Consider Y hyperelliptic surface of top type $t=(g,-,1)$; $Y=\mathbb{H}/\Gamma$ Γ S.Gr

Γ index 2 subgr in Δ; $s(\Delta)=(0;+;[2,\overset{g}{\ldots},2],\{(2,2)\})$

$Aut(Y)\leq C_2\times C_2$ (Bujalance-Etayo-Gamboa-Gromadzki, 1990)

Geometrically, we have the configuration for the action of $Aut(Y)/\langle\varphi\rangle$ (φ hyperelliptic involution)



each arc of the boundary setwise fixed

$\boxed{2r+s=g+3}$

If we have $\theta_r:\Lambda\longrightarrow C_2\times C_2=\langle a,b\rangle$ $0\leq r\leq[g/2]$

$s(\Lambda)=(0;+;[2^r];\{(2^s)\})$ and monodromies

$\theta_r:\Lambda\longrightarrow C_2\times C_2=\Lambda/\Gamma$; $s(\theta^{-1}\langle a\rangle)=(0;+;[2^g];\{(2,2)\})$

$\theta_r(x_i)=a$; $\theta_r(e)=a/\mathrm{Id}$ according r's parity

$\theta(c_0)=\theta(c_s)=a$; $\theta_r(c_1)=\mathrm{Id}$

Alternating $\theta_r(c_{2j})=b$; $\theta_r(c_{2j+1})=ab$

The actions given by θ_r are maximal. They produce $\frac{g}{2}+1=\frac{g+2}{2}$ connected components for g even and $\frac{g-1}{2}+1=\frac{g+1}{2}$ connected components for g odd

$\mathcal{B}^{k, Hyp}_{(g, t, 1)}$ is connected.

Consider again Y hyper. with top type $t = (g, t, 1)$, φ hyper. involution $Y = \mathbb{H}/\Gamma$ and $Y/\langle\varphi\rangle = \mathbb{H}/\Lambda$ with $s(\Delta) = (0; t; [2^{2g+1}]; \{(-)\})$

(a disc with $2g+1$ cone pts

The groups of automorphisms of $Y/\langle\varphi\rangle$ can be dihedral or cyclic

$\mathrm{Aut}(Y)$: $C_n \times C_2$, n a proper divisor of $2g+1$

$s(\Lambda) = (0; t; [n, 2^r], \{(-)\})$ $r = \frac{2g+1}{n}$

C_{2n} ; n a proper divisor of $2g$

$s(\Lambda) = (0; t; [2n, 2^r], \{(-)\})$; $r = 2g/n$

D_n ; n an even divisor of $4g$

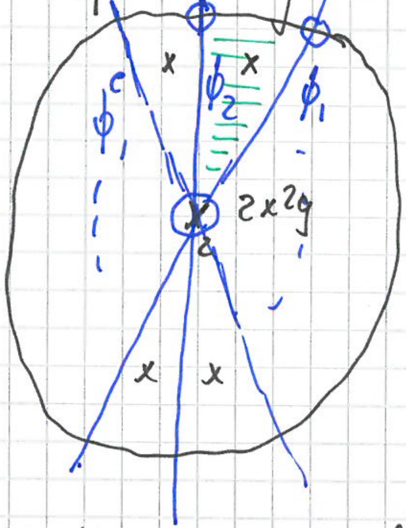
$s(\Lambda) = (0; t; [2^r]; \{(n; 2^{s-2})\}$; $s = \frac{4g}{n} + 2 - 2r$

$D_{n/2} \times C_2$; n an even divisor of $4g+2$

$s(\Lambda) = (0; t; [2^r]; \{(\frac{n}{2}, 2^s)\})$; $s = \frac{4g+2}{n} + 2 - 2r$

(Bujalance - Etayo - Gamboa - Gromadzki, 1990)

Graphically : Consider configurations



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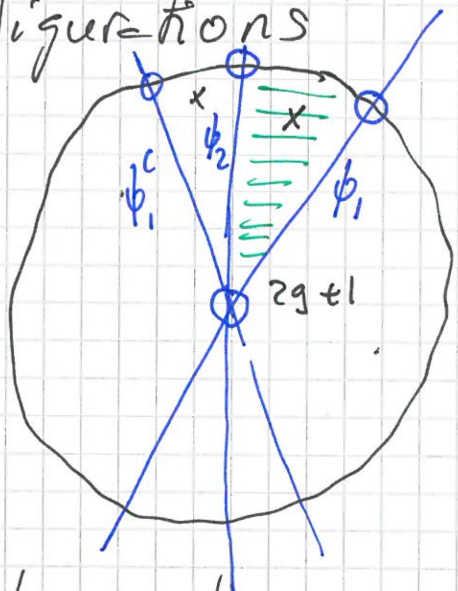
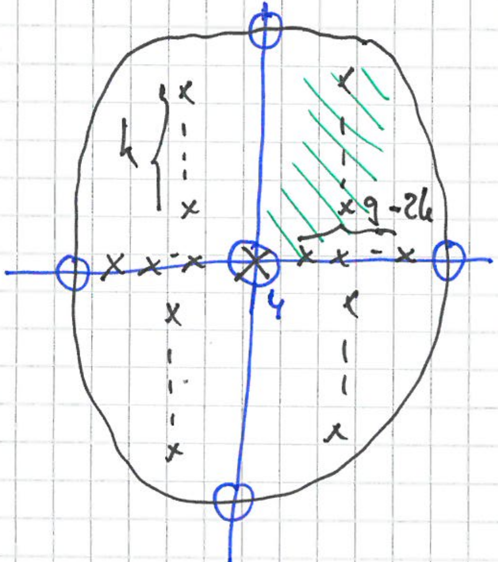
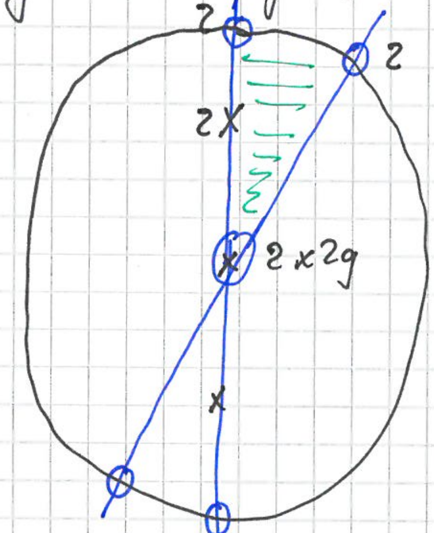


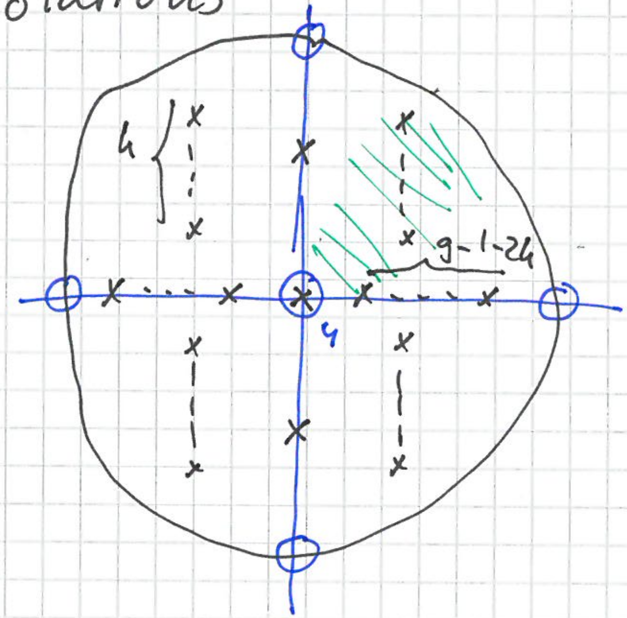
The connect rotations and anticonformal involution of top. type

$(0; t; [2, \overset{g}{-}-2], \{(2,2,2)\})$

The following configurations show actions connecting all the strata induced by anticonformal involutions







THANK YOU