Belyĭ Maps on Elliptic Curves and Dessin d'Enfants on the Torus

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Abstract

A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these values are $\{0, 1, \infty\}$. A Dessin d'Enfant is a planar bipartite graph obtained by considering the preimage of a path between two of these critical values, usually taken to be the line segment from 0 to 1. Such graphs can be drawn on the sphere by composing with stereographic projection: $\beta^{-1}([0,1]) \subseteq \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. Replacing $\mathbb{P}^1$ with an elliptic curve E, there is a similar definition of a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. The corresponding Dessin d'Enfant can be drawn on the torus by composing with an elliptic logarithm: $\beta^{-1}([0,1]) \subseteq E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$.

In this talk, we discuss the problems of (1) constructing examples of Belyĭ maps for elliptic curves and (2) drawing Dessins d'Enfants on the torus. This is work part of PRiME (Purdue Research in Mathematics Experience) with Leonardo Azopardo, Sofia Lyrintzis, Bronz McDaniels, Maxim Millan, Yesid Sánchez Arias, Danny Sweeney, and Sarah Thomaz with assistance by Hongshan Li and Avi Steiner.

Research Experiences for Undergraduate Faculty
June 4, 2012 – June 8, 2012

http://aimath.org/ARCC/workshops/reuf4.html

Edray Goins — Purdue University

Luis Melara — Shippensburg University

Karoline Null — American Association Advancement of Science

Alejandra Alvarado — Eastern Illinois University

Emille Lawrence — University of San Francisco

Naiomi Cameron — Lewis and Clark College

Roselyn Williams — Florida A&M University

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June 15, 2015 – August 7, 2015

Outline of Talk

Riemann Surfaces

A **Riemann Surface** is a triple $(X, \{U_\alpha\}, \{\mu_\alpha\})$ satisfying:

- **Coordinate Charts and Maps:** For some countable indexing set I,

$$X = \bigcup_{\alpha \in I} U_\alpha \qquad \text{and} \qquad \mu_\alpha : \ U_\alpha \hookrightarrow \mathbb{C}.$$

- **Locally Euclidean:** Each $\mu_\alpha(U_\alpha)$ is a connected, open subset of $\mathbb{C}$; and the composition $\mu_\beta \circ {\mu_\alpha}^{-1}$ is a smooth function.

$$\begin{array}{ccccc} \mathbb{C} & & X & & \mathbb{C} \\ \cup\!| & \xrightarrow{{\mu_\alpha}^{-1}} & \cup\!| & \xrightarrow{\mu_\beta} & \cup\!| \\ \mu_\alpha\left(U_\alpha \cap U_\beta\right) & & U_\alpha \cap U_\beta & & \mu_\beta\left(U_\alpha \cap U_\beta\right) \end{array}$$

- **Hausdorff:** For distinct $z \in U_\alpha$ and $w \in U_\beta$ there exist open subsets

$$\begin{array}{c} \mu_\alpha(z) \in \mathcal{U}_\alpha \subseteq \mu_\alpha(U_\alpha) \\ \mu_\beta(w) \in \mathcal{U}_\beta \subseteq \mu_\beta(U_\beta) \end{array} \quad \text{such that} \quad {\mu_\alpha}^{-1}(\mathcal{U}_\alpha) \cap {\mu_\beta}^{-1}(\mathcal{U}_\beta) = \emptyset.$$

Both $\mathbb{C} \cup \{\infty\}$ and $\mathbb{C}/\Lambda$ are examples. We always embed $X \hookrightarrow \mathbb{R}^3$.

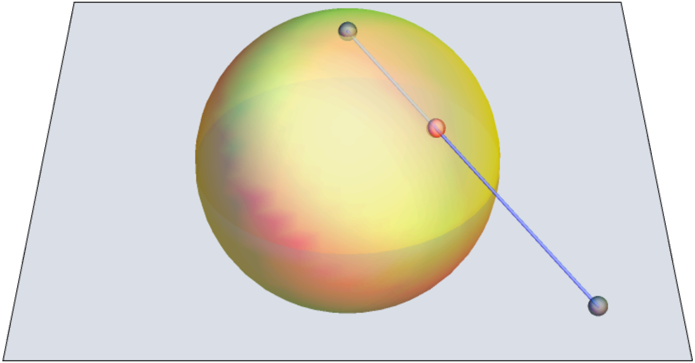
Elliptic Riemann Surfaces

Theorem

The complex plane $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ *is the same as the unit sphere*

$$S^2(\mathbb{R}) = \left\{ (u, v, w) \in \mathbb{R}^3 \,\middle|\, u^2 + v^2 + w^2 = 1 \right\}.$$

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \longrightarrow & S^2(\mathbb{R}) \\ z = \dfrac{u + i\,v}{1 - w} = \dfrac{1 + w}{u - i\,v} & \mapsto & (u, v, w) = \left(\dfrac{2\,\mathrm{Re}\, z}{|z|^2 + 1}, \dfrac{2\,\mathrm{Im}\, z}{|z|^2 + 1}, \dfrac{|z|^2 - 1}{|z|^2 + 1} \right) \\ 0, \quad 1, \quad \infty & \mapsto & (0, 0, -1), \quad (1, 0, 0), \quad (0, 0, 1) \end{array}$$



http://en.wikipedia.org/wiki/Stereographic_projection

Parabolic Riemann Surfaces

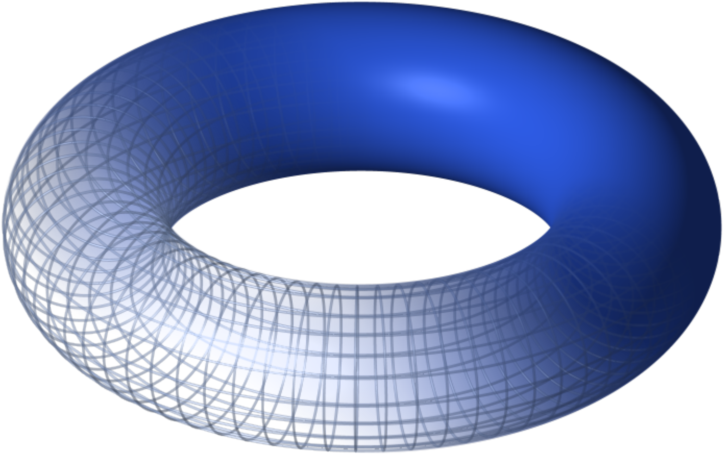
Theorem

Relative to a lattice Λ, the quotient space $\mathbb{C}/\Lambda$ is the same as the torus

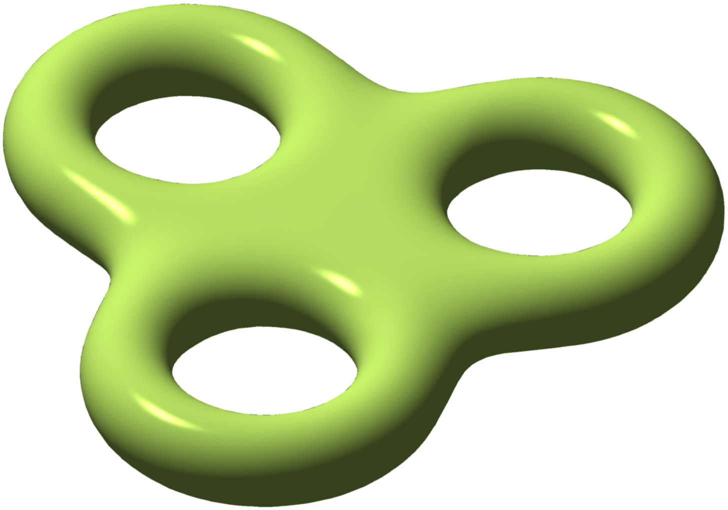
$$T^2(\mathbb{R}) = \left\{ (u, v, w) \in \mathbb{R}^3 \,\middle|\, \left(\sqrt{u^2 + v^2} - R\right)^2 + w^2 = r^2 \right\}.$$

$$\mathbb{C}/\Lambda \simeq (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \longrightarrow T^2(\mathbb{R})$$

$$\begin{aligned} \theta &= \arctan \frac{v}{u} \\ \phi &= \arctan \frac{w}{\sqrt{u^2 + v^2} - R} \end{aligned} \quad \mapsto \quad \begin{aligned} u &= \big(R + r \cos\theta\big) \cos\phi \\ v &= \big(R + r \cos\theta\big) \sin\phi \\ w &= r \sin\theta \end{aligned}$$



http://en.wikipedia.org/wiki/Torus



http://en.wikipedia.org/wiki/Genus_(mathematics)

Monodromy

Let $\beta : X \to \mathbb{P}^1(\mathbb{C})$ be a rational map of $\deg \beta = N$ on a compact, connected Riemann surface X.

- This is ramified over $S = \{w_0, w_1, \ldots, w_r\} \subseteq \mathbb{P}^1(\mathbb{C})$, so we have a degree N covering map from $X_S = X - \beta^{-1}(S)$ to $F_S^1 = \mathbb{P}^1(\mathbb{C}) - S$.
- For any $w \notin S$, the homology group is the free group of rank r:

$$\pi_1(F_S^1, w) = \{\gamma : [0,1] \to F_S^1 \mid \gamma(0) = \gamma(1) = w\} / \sim$$
$$= \left\langle \gamma_0, \gamma_1, \ldots, \gamma_r \;\middle|\; \gamma_0 * \gamma_1 * \cdots * \gamma_r = 1 \right\rangle .$$

- For each loop $\gamma : [0,1] \to F_S^1$ and point $z_k \in \beta^{-1}(w)$, there is a unique path $\widetilde{\gamma}_k : [0,1] \to X_S$ such that $\beta \circ \widetilde{\gamma}_k = \gamma$ and $z_k = \widetilde{\gamma}_k(0)$.

$$\begin{array}{ccccc} \pi_1\left(F_S^1, w\right) & \longrightarrow & \mathrm{Aut}\left(\beta^{-1}(w)\right) & \xrightarrow{\sim} & \mathrm{Sym}(N) \\ \gamma & & \left[z_k = \widetilde{\gamma}_k(0) \mapsto \widetilde{\gamma}_k(1) = z_{\sigma(k)}\right] & & \sigma \end{array}$$

- The **monodromy group of** β is the image generated by $\gamma_k \mapsto \sigma_k$:

$$G = \left\langle \sigma_0, \sigma_1, \ldots, \sigma_r \;\middle|\; \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_r = 1 \right\rangle \simeq \mathrm{Gal}\left(\mathbb{C}(X) / \beta^* \mathbb{C}(t)\right).$$

Theorem (Adolph Hurwitz, 1891)

Fix a compact, connected Riemann surface X of genus g. Let

$$\mathcal{D} = \Big\{ \{ e_P \,|\, P \in B_0 \}, \ \{ e_P \,|\, P \in B_1 \}, \ \ldots, \ \{ e_P \,|\, P \in B_r \} \Big\}$$

be a collection of $(r+1)$ partitions of N such that

$$2\,N = (2 - 2\,g) + \sum_{k=0}^{r} \left[\sum_{P \in B_k} (e_P - 1) \right].$$

Then $\mathcal{D}$ the branch data of a rational map $\beta : X \to \mathbb{P}^1(\mathbb{C})$ (that is, $\deg \beta = N$ and $B_k = \beta^{-1}(w_k)$ where e_P are the ramification indices) if and only if there exist $\sigma_0, \sigma_1, \ldots, \sigma_r \in S_N$ such that

- $\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_r = 1.$
- $G = \langle \sigma_0, \sigma_1, \ldots, \sigma_r \rangle$ *is a transitive subgroup of* S_N.
- $\sigma_k = \prod_{P \in B_k} \left(a_{k,1} \ a_{k,2} \ \cdots \ a_{k,e_P} \right)$ *for* $k = 0, 1, \ldots, r$.

Motivating Questions

- When $X \simeq \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$ has genus $g = 0$, what are examples of rational maps $\beta : X \to \mathbb{P}^1(\mathbb{C})$?
- When $X \simeq \mathbb{C}/\Lambda \simeq T^2(\mathbb{R})$ has genus $g = 1$, what are examples of rational maps $\beta : X \to \mathbb{P}^1(\mathbb{C})$?
- Given branch data $\mathcal{D}$, can we determine the monodromy group G?
- Which mondromy groups G appear? And can we use this to construct extensions $\mathbb{C}(X)$ of $\beta^*\mathbb{C}(t)$ having interesting Galois groups?

We will be interested in the case where $r = 2$, so that $\beta : X \to \mathbb{P}^1(\mathbb{C})$ is a **Belyĭ map**.

Alexander Grothendieck (March 28, 1928 – November 13, 2014)
http://en.wikipedia.org/wiki/Alexander_Grothendieck

Belyĭ Maps

Denote X as a Riemann Surface. We always embed $X \hookrightarrow \mathbb{R}^3$.

A **rational function** $\beta : X \to \mathbb{P}^1(\mathbb{C})$ is a map which is a ratio $\beta(z) = p(z)/q(z)$ in terms of relatively prime polynomials $p,\, q \in \mathbb{C}[X]$; define its **degree** as $\deg \beta = \max\left\{\deg p,\ \deg q\right\}$.

Theorem (Fundamental Theorem of Algebra)

For $w \in \mathbb{P}^1(\mathbb{C})$, denote $\beta^{-1}(w) = \left\{ z \in X \,\middle|\, p(z) - w\, q(z) = 0 \right\}$. Then $\left|\beta^{-1}(w)\right| \leq \deg \beta$.

- $w \in \mathbb{P}^1(\mathbb{C})$ is said to be a **ramification point** if $\left|\beta^{-1}(w)\right| \neq \deg \beta$.
- A **Belyĭ map** is a rational function β such that its collection of critical values w is contained within the set $S = \{0,\, 1,\, \infty\} \subseteq \mathbb{P}^1(\mathbb{C})$.

Dessins d'Enfant

Fix a Belyĭ map $\beta(z) = p(z)/q(z)$ for X. Denote the preimages

$$
\begin{aligned}
B &= \beta^{-1}(\{0\}) &=& \left\{ z \in X \,\middle|\, p(z) = 0 \right\} \\
W &= \beta^{-1}(\{1\}) &=& \left\{ z \in X \,\middle|\, p(z) - q(z) = 0 \right\} \\
F &= \beta^{-1}(\{\infty\}) &=& \left\{ z \in X \,\middle|\, q(z) = 0 \right\} \\
E &= \beta^{-1}([0,1]) &=& \left\{ z \in X \,\middle|\, \beta(z) \in \mathbb{R} \text{ and } 0 \leq \beta(z) \leq 1 \right\}
\end{aligned}
$$

The bipartite graph (V, E) with vertices $V = B \cup W$ and edges E is called **Dessin d'Enfant**. We embed the graph on X in 3-dimensions.

$$
\begin{array}{ccccc}
B = \beta^{-1}(0) & W = \beta^{-1}(1) & F = \beta^{-1}(\infty) & E = \bigcup\limits_{0 \leq t \leq 1} \beta^{-1}(t) & X \xrightarrow{J} \mathbb{P}^1(\mathbb{C}) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\left\{ \begin{array}{c} \text{black} \\ \text{vertices} \end{array} \right\} & \left\{ \begin{array}{c} \text{white} \\ \text{vertices} \end{array} \right\} & \left\{ \begin{array}{c} \text{midpoints} \\ \text{of faces} \end{array} \right\} & \left\{ \text{edges} \right\} & \mathbb{R}^3
\end{array}
$$

Properties of Dessins

Theorem

Let $\beta : X \to \mathbb{P}^1(\mathbb{C})$ be a Belyĭ map on a Riemann Surface X. Define a bipartite graph $(B \cup W, E)$ by choosing

$$B = \beta^{-1}(\{0\}), \quad W = \beta^{-1}(\{1\}), \quad F = \beta^{-1}(\{\infty\}), \quad E = \beta^{-1}([0,1]).$$

- *The graph can be embedded in X without crossings.*
- *The number of vertices, edges, and faces is*

$$v = \left|\beta^{-1}(\{0,\,1\})\right| \qquad e = \deg \beta \qquad f = \left|\beta^{-1}(\{\infty\})\right|.$$

- *We have $v - e + f = 2 - 2\,g$ in terms of the genus*

$$g = 1 + \frac{1}{2}\left[\deg \beta - \left|\beta^{-1}(\{0,\,1,\,\infty\})\right|\right].$$

Belyĭ Maps on the Riemann Sphere

Denote $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$ as the Riemann Sphere.

$$G = Z_n : \quad \beta(z) = z^n$$

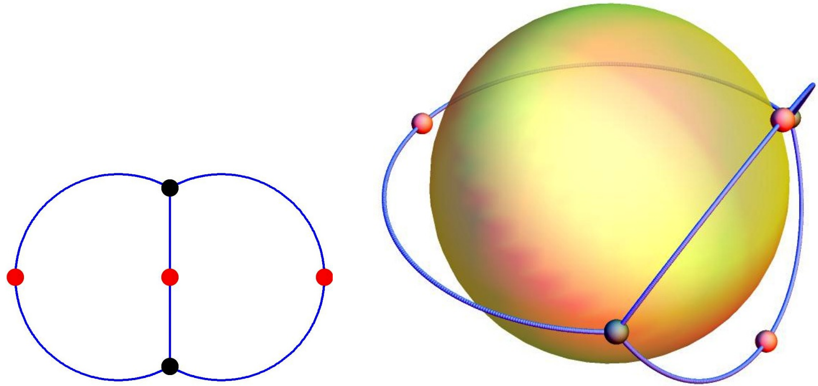
$$G = D_n : \quad \beta(z) = \frac{4\, z^n}{\left(z^n + 1\right)^2}$$

$$G = S_3 : \quad \beta(z) = \frac{4}{27} \frac{\left(z^2 - z + 1\right)^3}{z^2 \left(z - 1\right)^2}$$

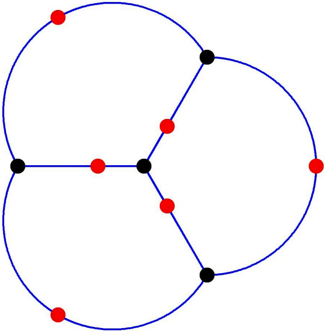
$$G = A_4 : \quad \beta(z) = \frac{\left(z^4 + 2\sqrt{2}\, z\right)^3}{\left(2\sqrt{2}\, z^3 - 1\right)^3}$$

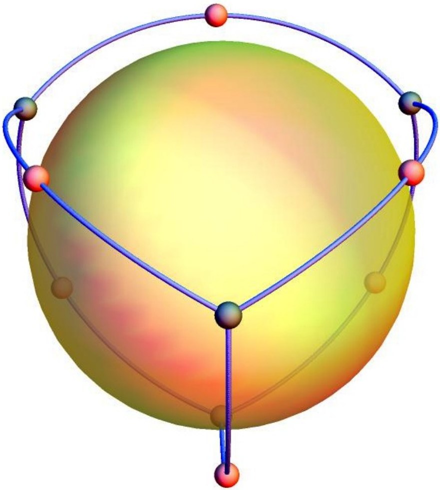
$$G = S_4 : \quad \beta(z) = \frac{1}{108} \frac{\left(z^8 + 14\, z^4 + 1\right)^3}{z^4 \left(z^4 - 1\right)^4}$$

$$G = A_5 : \quad \beta(z) = \frac{1}{1728} \frac{\left(z^{20} + 228\, z^{15} + 494\, z^{10} - 228\, z^5 + 1\right)^3}{z^5 \left(z^{10} - 11\, z^5 - 1\right)^5}$$

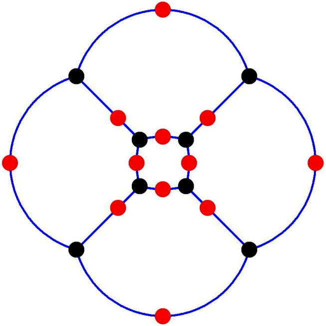


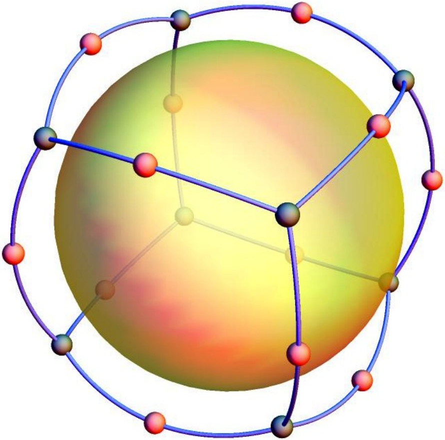
$$\beta(z) = \frac{4}{27}\,\frac{\left(z^2 - z + 1\right)^3}{z^2\left(z-1\right)^2}: \qquad v = 5, \quad e = 6, \quad f = 3$$



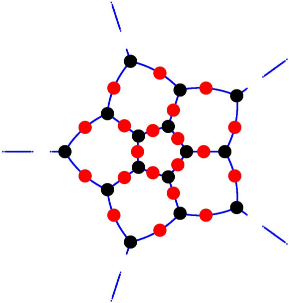


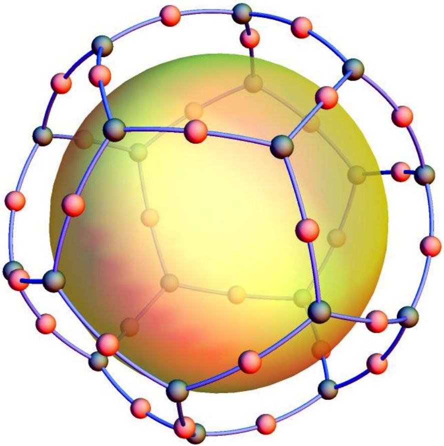
$$\beta(z) = \frac{\left(z^4 + 2\sqrt{2}\,z\right)^3}{\left(2\sqrt{2}\,z^3 - 1\right)^3} : \qquad v = 10, \quad e = 12, \quad f = 4$$





$$\beta(z) = \frac{1}{108} \, \frac{\left(z^8 + 14\, z^4 + 1\right)^3}{z^4 \left(z^4 - 1\right)^4} : \qquad v = 20, \quad e = 24, \quad f = 6$$





$$\beta(z) = \frac{1}{1728} \frac{\left(z^{20}+228\,z^{15}+494\,z^{10}-228\,z^{5}+1\right)^{3}}{z^{5}\left(z^{10}-11\,z^{5}-1\right)^{5}} : \quad v = 50, \quad e = 60, \quad f = 12$$

Elliptic Curves

Theorem

$$X = \left\{ (x,y) \in \mathbb{C}^2 \,\middle|\, y^2 = x^3 + A\,x + B \right\} \simeq T^2(\mathbb{R}) \text{ for complex } A \text{ and } B.$$

With e_1, e_2, and e_3 as distinct complex roots of $x^3 + A\,x + B = 0$, write the **period lattice** $\Lambda = \mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$ in terms of the **elliptic integrals**

$$w_1 = \frac{1}{\pi} \int_{e_1}^{e_3} \frac{dz}{\sqrt{z^3 + A\,z + B}} \quad \text{and} \quad w_2 = \frac{1}{\pi} \int_{e_2}^{e_3} \frac{dz}{\sqrt{z^3 + A\,z + B}}.$$

Define the maps

$$\begin{array}{ccccc} X & \longrightarrow & \mathbb{C}/\Lambda & \longrightarrow & T^2(\mathbb{R}) \\ (x,y) & \mapsto & \begin{aligned} &\theta\,w_1 + \phi\,w_2 \\ &= \int_{e_1}^{x} \frac{dz}{\sqrt{z^3 + A\,z + B}} \end{aligned} & \mapsto & \begin{aligned} u &= \big(R + r\,\cos\theta\big)\,\cos\phi \\ v &= \big(R + r\,\cos\theta\big)\,\sin\phi \\ w &= r\,\sin\theta \end{aligned} \end{array}$$

Belyĭ Maps on Elliptic Curves

$$E: \; y^2 = x^3 + 1 \qquad \beta(x,y) = \frac{y+1}{2}$$

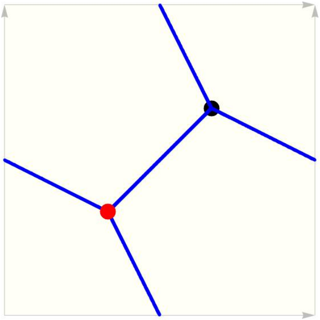
$$E: \; y^2 = x^3 + 5\,x + 10 \qquad \beta(x,y) = \frac{(x-5)\,y + 16}{32}$$

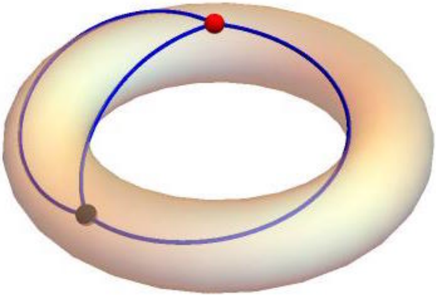
$$E: \; y^2 = x^3 - 120\,x + 740 \qquad \beta(x,y) = \frac{(x+5)\,y + 162}{324}$$

$$E: \; y^2 + x\,y + y = x^3 + x^2 + 35\,x - 28 \qquad \beta(x,y) = \frac{4\left(9\,x\,y - x^3 - 15\,x^2 - 36\,x + 32\right)}{3125}$$

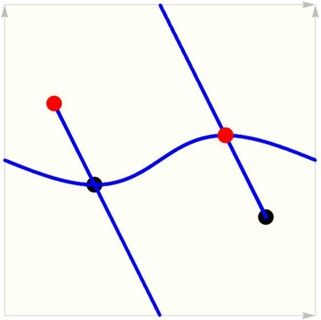
$$E: \; y^2 = x^3 - 15\,x - 10 \qquad \beta(x,y) = \frac{\left(3\,x^2 + 12\,x + 5\right) y + \left(-10\,x^3 - 30\,x^2 - 6\,x + 6\right)}{-16\left(9\,x + 26\right)}$$

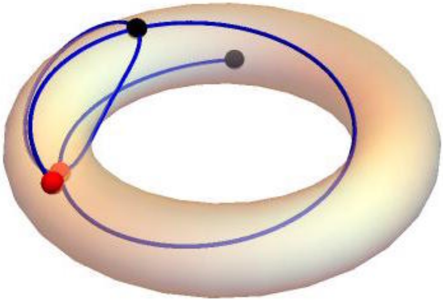
$$E: \; y^2 + 15\,x\,y + 128\,y = x^3 \qquad \beta(x,y) = \frac{\left(y - x^2 - 17\,x\right)^3}{16384\,y}$$



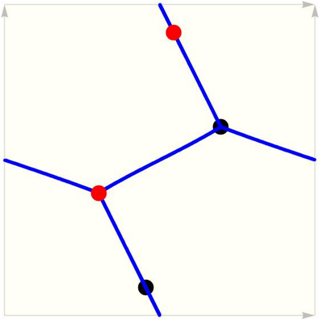


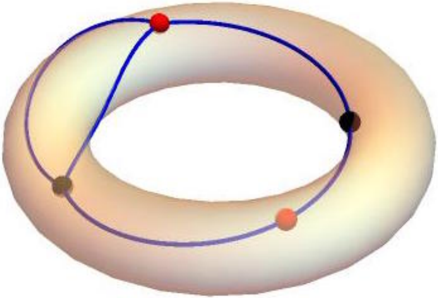
$$\beta(x,y) = \frac{y+1}{2} \qquad \text{on} \qquad E:\; y^2 = x^3 + 1$$



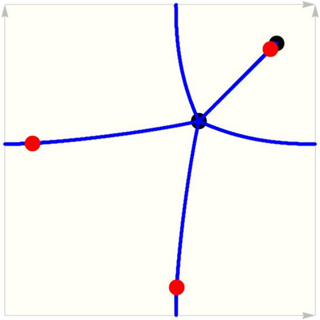


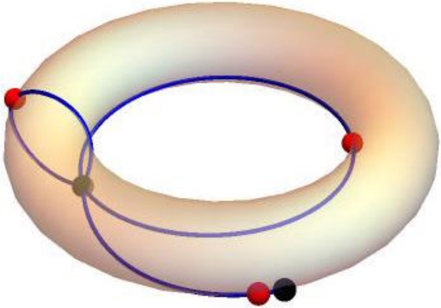
$$\beta(x,y) = \frac{(x-5)\,y+16}{32} \qquad \text{on} \qquad E:\; y^2 = x^3 + 5\,x + 10$$





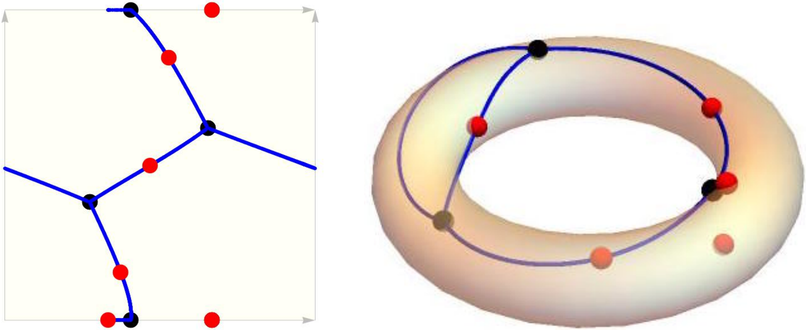
$$\beta(x, y) = \frac{(x+5)\,y + 162}{324} \qquad \text{on} \qquad E: \; y^2 = x^3 - 120\,x + 740$$





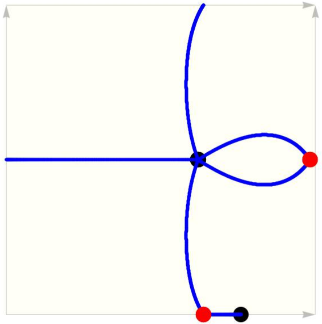
$$\beta(x,y) = \frac{4\left(9\,x\,y - x^3 - 15\,x^2 - 36\,x + 32\right)}{3125} \text{ on}$$

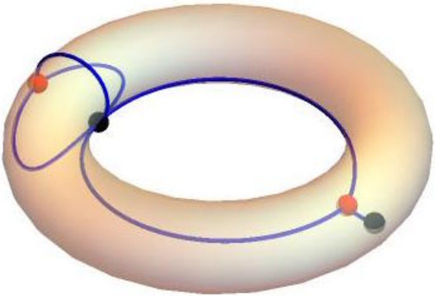
$$E: \; y^2 + x\,y + y = x^3 + x^2 + 35\,x - 28$$



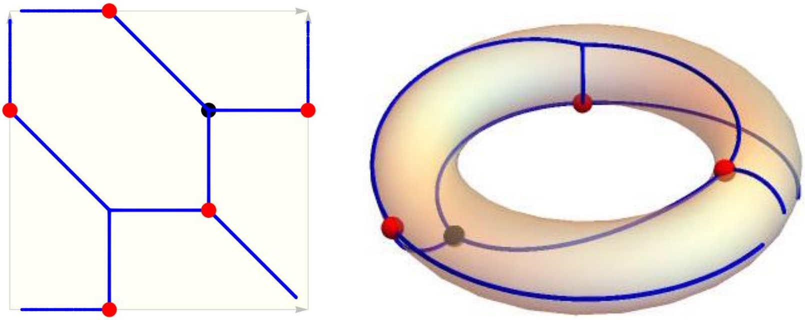
$$\beta(x,y) = \frac{\left(3\,x^2+12\,x+5\right)\,y + \left(-10\,x^3-30\,x^2-6\,x+6\right)}{-16\,\left(9\,x+26\right)} \quad \text{on}$$

$$E: \;\; y^2 = x^3 - 15\,x - 10$$





$$\beta(x,y) = \frac{\left(y - x^2 - 17\,x\right)^3}{16384\,y} \qquad \text{on} \qquad E:\; y^2 + 15\,x\,y + 128\,y = x^3$$



$$\beta(x,y) = \frac{216\,x^3}{(y+36)^3} \qquad \text{on} \qquad E:\ y^2 = x^3 - 432$$

Thank You!

Questions?