# Visualizing Dessins d'Enfants on the Torus

**PRiME 2015**
Department of Mathematics
Purdue University

Summer Project Presentation
Purdue University

October 04, 2015

# Overview of the algorithm

Input:

- An elliptic curve in the form:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

- A Belyĭ Map: $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$

Output: Points and coloring corresponding to a Dessin d'Enfant embedded on the torus, $\mathbb{T}^2(\mathbb{R})$.

# Step 1: Setup

We find a set of points $(x, y)$ that approximate $\beta^{-1}$ on the curve from 0 to 1: $\beta^{-1}([0,1]) \subseteq E(\mathbb{C})$.

Recall:

- $\beta^{-1}(0) =$ Red Vertices
- $\beta^{-1}(1) =$ Black Vertices
- $\beta^{-1}([0,1]) =$ Edges

# Step 1: Setup

Given our curve :

$$y^2 + a_1\, x\, y + a_3, y = x^3 + a_2\, x^2 + a_4\, x + a_6$$

We simplify it to the following

$$4(x^3 + a_2 x^2 + a_4 x + a6) + (a_1 x + a_3)^3 = 4(x - e_1)(x - e_2)(x - e_3)$$

Where $e_1$, $e_2$, and $e_3$ are roots of the elliptic curve

# Step 2: The elliptic logarithm

Recall the lattice $\Lambda = \left\{ m\,\omega_1 + n\,\omega_2 \,\middle|\, m,\ n \in \mathbb{Z} \right\}$ in terms of the periods

$$\omega_1 = \int_{e_1}^{e_3} \frac{dt}{\sqrt{(t-e_1)\,(t-e_2)\,(t-e_3)}} \qquad \text{and} \qquad \omega_2 = \int_{e_2}^{e_3} \frac{dt}{\sqrt{(t-e_1)\,(t-e_2)\,(t-e_3)}}$$

For every point $(x, y)$ calculated in step 1 we compute an elliptic logarithm, z by $\log_E : E(\mathbb{C}) \to \mathbb{C}/\Lambda$.:

$$\begin{aligned} \mathrm{z} &= sgn(y) \int_{x}^{\infty} \frac{dt}{\sqrt{(t-e_1)\,(t-e_2)\,(t-e_3)}} \\ &= m\,\omega_1 + n\,\omega_2 \end{aligned}$$

# Step 3: Projecting to the torii

We exploit the following isomorphisms

$$E(\mathbb{C}) \xrightarrow{\simeq} \mathbb{C}/\Lambda \xrightarrow{\simeq} T^2(\mathbb{R})$$

$$(x, y) \qquad \operatorname{sgn}(y) \int_x^{\infty} \frac{dt}{\sqrt{(t-e_1)\,(t-e_2)\,(t-e_3)}} = m\,\omega_1 + n\,\omega_2$$

$$u = \Big(R + r\,\cos 2\pi m\Big)\,\cos 2\pi n$$

$$v = \Big(R + r\,\cos 2\pi m\Big)\,\sin 2\pi n$$

$$w = r\,\sin 2\pi n$$

# John Cremona and Thotasaphon Thongjunthug's Method

- One of the major issues was calculating the elliptic logarithm in Sage.
- We used the Arithmetic Geometric Mean(AGM) to approximate the elliptic integral.
- Without using an integral within a few steps you can acquire the elliptic integral quickly.

# Overview of the Algorithm

- The algorithm is broken up into two steps
  1. Finding the periods $\omega_1$ and $\omega_2$
  2. Finding the elliptic logarithm
- To find the periods you set up one "for" loop that converges to $\omega_1$ and $\omega_2$ as you increase the iterations
- To find the elliptic logarithm you set up a nested "for" loop that calculates a complex number for every point on the elliptic curve.

# Step 1 - Calculating the Periods

- To begin calculate four values in terms of the roots of the elliptic curve.

$$A_0 = \sqrt{e_1 - e_3} \quad B_0 = \sqrt{e_1 - e_2} \quad C_0 = \sqrt{e_2 - e_3} \quad D_0 = \sqrt{e_2 - e_1}$$

- Next we run through the "for" loop from $p = 0, 1, 2, 3 \ldots N - 1$

$$A_{p+1} = \frac{A_p + B_p}{2} \quad B_{p+1} = \sqrt{A_p B_p} \quad C_{p+1} = \frac{C_p + D_p}{2} \quad D_{p+1} = \sqrt{C_p D_p}$$

- As p iterates to N-1 the following ratio converges to the period

$$\omega_1 = \frac{\pi}{A_N} = \frac{\pi}{B_N} \qquad \omega_2 = \frac{\pi}{C_N} = \frac{\pi}{D_N}$$

Remark: There is a consistent way to choose the signs to guarantee convergence

# Step 2 - Calculating the Elliptic Logarithm

Given a point $(x, y)$ on the elliptic curve, compute the following

- To calculate the elliptic logarithm find the following values

$$I_1 = \sqrt{\frac{x - e_1}{x - e_2}} \qquad J_1 = \frac{-(2y + a_1 x + a_3)}{2 I_1 (x - e_2)}$$

- Then as $p = 0, 1, 2, 3, 4 \ldots N - 1$ calculate the following values

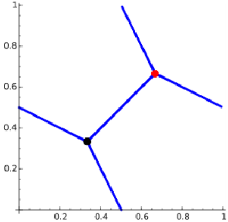
$$I_{p+1} = \sqrt{\frac{A_p (I_p + 1)}{B_{p-1} I_p + A_{p-1}}} \qquad J_{p+1} = I_{p+1} J_p$$

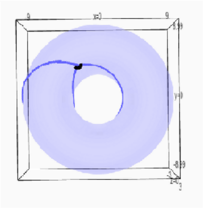
- This returns the elliptic logarithm

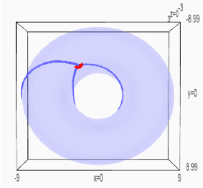
$$z = \frac{arctan \frac{A_N}{J_N}}{A_N}$$

# Dessin of Degree 3

$$E : y^2 = x^3 + 1 \qquad \beta(x, y) = \frac{(y+1)}{2}$$



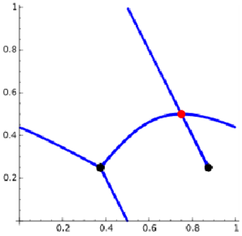



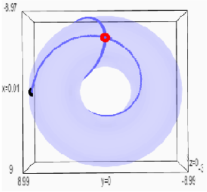


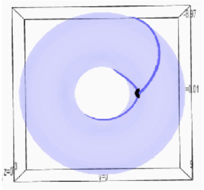
# Dessin of Degree 4

$$E : y^2 = x^3 + x^2 + 16x + 180$$

$$\beta(x, y) = \frac{(x^2+4y+56)}{(108)}$$

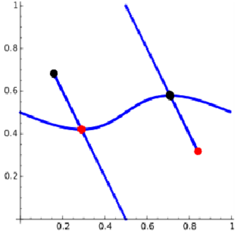


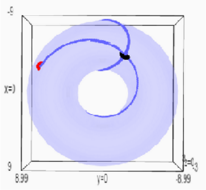



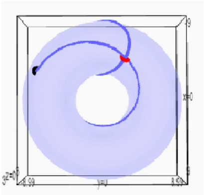


# Dessin of Degree 5

$$E : y^2 = x^3 + 5x + 10 \qquad \beta(x, y) = \frac{(x-5)y+16}{32}$$

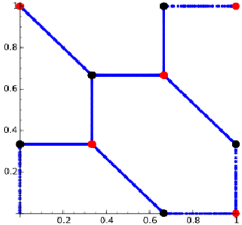


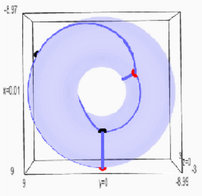



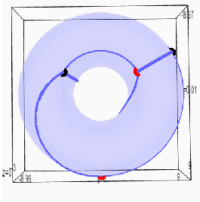


# Dessin of Degree 9

$$E : y^2 = x^3 - 432 \qquad \beta(x,y) = \frac{(216x^3)}{(y+36)^3}$$





# Acknowledgments


- Dr. Edray Herber Goins
- Hongshan Li
- Avi Steiner

- Dr. Steve Bell
- Dr. David Goldberg
- Dr. Lazslo Lempert
- Dr. Joel and Mrs. Ruth Spira
- Dr. Uli Walther
- Dr. Mark Ward
- Dr. Lowell W. Beineke

- Dr. Gregery Buzzard / Department of Mathematics
- College of Science
- UREP-C / Universidad Nacional de Colombia
- National Science Foundation

# Thank You!

# Questions?