Examples of Belyĭ Maps for Elliptic Curves

PRiME 2015
Department of Mathematics
Purdue University

Summer Project Presentation
Purdue University

October 04, 2015

PURDUE UNIVERSITY

Riemann-Hurwitz Formula

Riemann-Hurwitz Genus Formula

Let $\beta : X \to Y$ be a rational map between Riemann surfaces. The Euler characteristics $\chi(X) = 2 - 2\,g_X$ and $\chi(Y) = 2 - 2\,g_Y$ are related by

$$\chi(X) = \deg(\beta) \cdot \chi(Y) - \sum_{p \in X} (e_p - 1)$$

for some positive integers e_p called the ramification indices.

As a special case:

$$X = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R}) : \qquad g_X = 1 \qquad \chi(X) = 0$$

$$Y = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R}) : \qquad g_Y = 0 \qquad \chi(Y) = 2$$

Belyĭ Maps

Let E be an elliptic curve.
A **rational function** $\beta : E(\mathbb{C}) \hookrightarrow \mathbb{P}^1(\mathbb{C})$ is a map which is a ratio $\beta(x, y, z) = p(x, y, z)/q(x, y, z)$ in terms of relatively prime homogeneous polynomials $p(x, y, z), q(x, y, z) \in \mathbb{C}[x, y, z]$.
Define its **degree** as the natural number

$$\deg(\beta) = \max_{\omega \in \mathbb{P}^1(\mathbb{C})} \left| \left\{ P \in E(\mathbb{C}) \,\middle|\, \beta(P) = \omega \right\} \right|$$

- $\omega \in \mathbb{P}^1(\mathbb{C})$ is said to be a **critical value** if $|\beta^{-1}(\omega)| \neq \deg(\beta)$.
- A **Belyĭ map** is a rational function β such that its collection of critical values ω is contained within the set $\{(0:1), (1:1), (1,0)\} \subseteq \mathbb{P}^1(\mathbb{C})$

Generating examples

Proposition

Every elliptic curve has degree 2, 3 rational maps f with 4 critical values say $\{(0:1),\ (1:1),\ (1:0),\ (\omega_0:1)\}$

Corollary

Say $\omega_0 = \dfrac{p}{q}$ is a rational number. Then the composition

$$\beta : E(\mathbb{C}) \xrightarrow{\ f\ } \mathbb{P}^1(\mathbb{C}) \xrightarrow{\ h\ } \mathbb{P}^1(\mathbb{C})$$

in terms of $h(\omega) = \omega^{p-q}(\omega-1)^{-p}\left(\omega - \dfrac{p}{q}\right)^{q}$ is a Belyĭ map of $\deg(\beta) = \deg(f)\cdot\max\{|p|,|q|\}$

Degree 2 rational map

Proposition

Given an elliptic curve $E' : y^2 z = (x - e_1 z)\,(x - e_2 z)\,(x - e_3 z)$, we have that a corresponding rational map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of $\deg(\beta) = 2$ with critical values $\{(0:1), (1:1), (1:0), (\omega_0:1)\}$ is given by

$$\beta(x, y, z) = \frac{e_2 - e_3}{e_2 - e_1} \cdot \frac{x - e_1 z}{x - e_3 z}$$

Degree 3 rational map

Every elliptic curve can be written in the form $y^2z + A_1xyz + A_3yz^2 = x^3$ for complex numbers A_1 and A_3.

Theorem

Given an elliptic curve $E : y^2z + A_1xyz + A_3yz^2 = x^3$, we have that a corresponding rational map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of $\deg(\beta) = 3$ with critical values $\{(0:1), (1:1), (1:0), (\omega_0:1)\}$ is given by

$$\beta(x,y,z) = \frac{\left(2\,A_1^3 - 27\,A_3 - 2\,A_1\sqrt{A_1\left(A_1^3 - 27\,A_3\right)}\right)y - 27\,A_3^2 z}{\left(2\,A_1^3 - 27\,A_3 + 2\,A_1\sqrt{A_1\left(A_1^3 - 27\,A_3\right)}\right)y - 27\,A_3^2 z} \quad \text{if } A_1 \neq 0.$$

Dessins d'Enfant

Fix a Belyĭ map $\beta(x, y, z) = p(x, y, z)/q(x, y, z)$ for $E(\mathbb{C})$. Denote the preimages

$$\begin{aligned} B &= \beta^{-1}((0:1)) &= \{P \in E(\mathbb{C}) | p(P) = 0\} \\ W &= \beta^{-1}((1:1)) &= \{P \in E(\mathbb{C}) | p(P) - q(P) = 0\} \\ E &= \beta^{-1}([0,1]) &= \{P \in E(\mathbb{C}) | \beta(P) \in \mathbb{R} \text{ and } 0 \le \beta(P) \le 1\} \end{aligned}$$

The bipartite graph (V, E) with vertices $V = B \cup W$ and edges E is called **Dessins d'Enfant**. We embed the graph on $E(\mathbb{C})$ in $\mathbb{R}^3$.

$$\begin{array}{ccccc} B & W & E & E(\mathbb{C}) \xrightarrow{\beta} \mathbb{P}^1(\mathbb{C}) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \left\{ \begin{array}{c} \text{black} \\ \text{vertices} \end{array} \right\} & \left\{ \begin{array}{c} \text{white} \\ \text{vertices} \end{array} \right\} & \left\{ \text{edges} \right\} & \mathbb{R}^3 \end{array}$$

Degree Sequences

Proposition

Let $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a Belyĭ map. Then the sets

$$\begin{aligned} &\left\{e_P \,\middle|\, P \in B\right\} & & B = \beta^{-1}\big((0:1)\big) \\ &\left\{e_P \,\middle|\, P \in W\right\} & \text{in terms of} \quad & W = \beta^{-1}\big((1:1)\big) \\ &\left\{e_P \,\middle|\, P \in F\right\} & & F = \beta^{-1}\big((1:0)\big) \end{aligned}$$

are each integer partitions of $\deg(\beta)$ such that

$$2\deg(\beta) = \sum_{P \in B}\big(e_P - 1\big) + \sum_{P \in W}\big(e_P - 1\big) + \sum_{P \in F}\big(e_P - 1\big).$$

These integer partitions are called the degree sequences of the Belyĭ map.

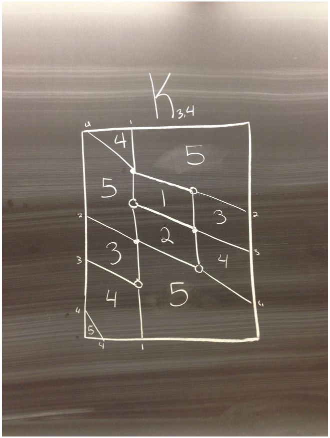
Unique degree sequences of Complete Bipartite Graphs

The only $K_{m,n}$ that can be drawn on the torus without edges crossing are $K_{3,3}$, $K_{3,4}$, $K_{3,5}$, $K_{3,6}$, $K_{4,4}$ (Ringel's Theorem). Recall Euler's formula

$$|B| = m, \quad |W| = n, \quad |E| = m\,n, \quad \text{and} \quad |F| = mn - (m+n).$$

$K_{m,n}$	$\lvert B\rvert$	$\lvert W\rvert$	$\lvert F\rvert$	Degree Sequence
$K_{3,3}$	3	3	3	$\{\{3,3,3\}\{3,3,3\},\{3,3,3\}\}$
$K_{3,4}$	3	4	5	$\{\{4,4,4\}\{3,3,3,3\},\{4,2,2,2,2\}\}$
$K_{3,5}$	4	5	7	$\{\{5,5,5\}\{3,3,3,3,3\},\{3,2,2,2,2,2,2\}\}$
$K_{3,6}$	5	6	9	$\{\{6,6,6\}\{3,3,3,3,3,3\},\{2,2,2,2,2,2,2,2,2\}\}$
$K_{4,4}$	4	4	8	$\{\{4,5,5\}\{3,3,3,3,3\},\{2,2,2,2,2,2,2,2\}\}$

$K_{3,4} : \{\{4, 4, 4\}\{3, 3, 3, 3\}, \{4, 2, 2, 2, 2\}\}$



Examples of Belyĭ Maps on Noam Elkies Webpage

Noam Elkies of Harvard University maintains a webpage which lists several examples of Belyĭ maps:
`http://www.math.harvard.edu/~elkies/nature.html`.
The following examples appear on the paper:

$$E: \; y^2 = x^3 + 1 \qquad \beta(x,y) = \frac{y+1}{2}$$

$$E: \; y^2 = x^3 + 5\,x + 10 \qquad \beta(x,y) = \frac{(x-5)\,y + 16}{32}$$

$$E: \; y^2 = x^3 - 120\,x + 740 \qquad \beta(x,y) = \frac{(x+5)\,y + 162}{324}$$

$$E: \; y^2 + 15\,x\,y + 128\,y = x^3 \qquad \beta(x,y) = \frac{\left(y - x^2 - 17\,x\right)^3}{2^{14}\,y}$$

Examples of Belyĭ Maps on Noam Elkies' Webpage

$$E : y^2 + x\,y + y = x^3 + x^2 + 35\,x - 28$$

$$\beta(x, y) = \frac{4\left(9\,x\,y - x^3 - 15\,x^2 - 36\,x + 32\right)}{3125}$$

And

$$E : y^2 = x^3 - 15\,x - 10$$

$$\beta(x, y) = \frac{\left(3\,x^2 + 12\,x + 5\right)y + \left(-10\,x^3 - 30\,x^2 - 6\,x + 6\right)}{-16\left(9\,x + 26\right)}$$

Examples of Belyĭ Maps on different papers

In Leonardo Zapponi's "On the Belyĭ degree(s) of a curve defined over a number field" (although the term "6 Y" should be "12 Y") there is the following example of a Belyĭ map:

$$E: \; y^2 = x^3 + x^2 + 16\,x + 180 \quad \beta(x,y) = \frac{x^2 + 4\,y + 56}{108}$$

In Lily Khadjavi and Victor Scharaschkin's "Belyĭ Maps, Elliptic Curves and the *ABC* Conjecture" one can find this other example of a Belyĭ map:

$$E: \; y^2 = x^3 - x \qquad \beta(x,y) = x^2$$

Verifying a rational function is a Belyĭ Map

Given an elliptic curve:

$$E(\mathbb{C}) = \left\{ (x, y) \in \mathbb{C}^2 \,\middle|\, y^2 + a_1\, x\, y + a_3\, y = x^3 + a_2\, x^2 + a_4\, x + a_6 \right\}$$

We verify that $\beta(x, y)$ is a Belyĭ map for $E(\mathbb{C})$ following the next steps

- Compute the maps degree.
  - Name $f(x, y) = y^2 + a_1\,x\,y + a_3\,y - (x^3 + a_2\,x^2 + a_4\,x + a_6)$ and solve $f(x, y) = 0$, $\beta(x, y) - \omega = 0$ (for any $\omega \in \mathbb{P}^1(\mathbb{C})$) in terms of x and y.
  - Compute the number of possible solutions and name that number the degree of the map.
- Compute a potential list of critical values.
  - Name $\overline{f}(y) = f(x, y)$, $\overline{\beta}(y) = \beta(x, y) - \omega$ and take the resultant of $\overline{f}$ and $\overline{\beta}$ because

Theorem

There exist $y \in \mathbb{C}$ satisfying $f(y) = g(y) = 0$ if and only if $Res(f, g) = 0$

  - Name $h(x) = Res(f, g)$ and check for which x it has repeated roots by looking when its discriminant vanishes.
- Determine which of the potential critical values are critical values.
  - Compute the number of elements in the inverse image of $\omega = \beta(x, y)$ for x such that $h(x) = 0$.

Degree Sequences of Belyĭ maps

By the Degree sequences proposition we have that:

- There do not exist any Belyĭ maps of $\deg(\beta) = 2$.
- Every Belyĭ map of $\deg(\beta) = 3$ or 4 has one of the following degree sequences, as illustrated by the accompanying examples:

$\{\{3\}, \{3\}, \{3\}\}$	$E: \; y^2 = x^3 + 1$	$\beta(x,y) = \dfrac{y+1}{2}$
$\{\{1, 3\}, \{4\}, \{4\}\}$	$E: \; y^2 = x^3 + x^2 + 16\,x + 180$	$\beta(x,y) = \dfrac{x^2 + 4\,y + 56}{108}$
$\{\{2, 2\}, \{4\}, \{4\}\}$	$E: \; y^2 = x^3 - x$	$\beta(x,y) = x^2$

Degree Sequences of Belyĭ maps

- Every Belyĭ map of $\deg(\beta) = 5$ has one of the following degree sequences, as illustrated by the accompanying examples:

$\{\{1,4\},\{1,4\},\{5\}\}$	$E : y^2 = x^3 + 5x + 10$	$\beta(x,y) = \frac{(x-5)\,y + 16}{32}$
$\{\{2,3\},\{2,3\},\{5\}\}$	$E : y^2 = x^3 - 120x + 740$	$\beta(x,y) = \frac{(x+5)\,y + 162}{324}$
$\{\{1,4\},\{2,3\},\{5\}\}$	?	?
$\{\{1,2,2\},\{5\},\{5\}\}$	?	?
$\{\{1,1,3\},\{5\},\{5\}\}$	?	?

Belyĭ Maps cant have Degree 2

Riemann Hurwitz Genus Formula for Belyĭ Maps on the torus

$$2\deg(\beta) = \sum_{P\in\beta^{-1}(0)} (e_P - 1) + \sum_{P\in\beta^{-1}(1)} (e_P - 1) + \sum_{P\in\beta^{-1}(\infty)} (e_P - 1)$$

- Critical Point when $e_P = 2$.
- $4 = (2-1) + (2-1) + (2-1) + (2-1)$
- 4 critical points $\Rightarrow$ 4 critical values
- Not a Belyĭ map.

Degree 3 Belyĭ Map

Set $E : y^2 = x^3 + Ax + B$ and $\beta(x, y) = \dfrac{ay + bx + c}{dy + ex + f} = \omega$

- Solve for y in terms of x and substitute into E.
- Factor $p(x) = (x - x_1)^{e_1} \cdots (x - x_n)^{e_n}$.
- Define y_k for $\beta(x_k, y_k) = \omega$ and $e_p = e_k$ for $P = (x_k, y_k)$.
- Have repeated roots when $\omega = 0, 1, \infty$.

$$2 \deg(\beta) = \sum_{P \in \beta^{-1}(0)} (e_P - 1) + \sum_{P \in \beta^{-1}(1)} (e_P - 1) + \sum_{P \in \beta^{-1}(\infty)} (e_P - 1)$$

- $\omega = 0, 1, \infty$:

$$p(x) = (x - x_1)^3 \Rightarrow e_1 = 3$$

- $\{\{3\}, \{3\}, \{3\}\}$

Database

Theorem (Zapponi)

There are only finitely many $\overline{\mathbb{Q}}$-isomorphism classes of elliptic curves of a given bounded Belyĭ degree.

Corollary

For a given positive integer N, there exists only finitely many $j_0 \in \mathbb{C}$ such that there exists an elliptic curve E and a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ with $j(E) = j_0$ and $\deg(\beta) = N$.

Database

Given an integer N, we want to compute

- Elliptic curves E.
- Belyĭ maps $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ with $\deg(\beta) = N$.
- Dessin d'Enfant of β

Acknowledgments

- Dr. Edray Herber Goins
- Hongshan Li
- Avi Steiner

- Dr. Steve Bell
- Dr. David Goldberg
- Dr. Lazslo Lempert
- Dr. Joel and Mrs. Ruth Spira
- Dr. Uli Walther
- Dr. Mark Ward
- Dr. Lowell W. Beineke

- Dr. Gregery Buzzard / Department of Mathematics
- College of Science
- UREP-C / Universidad Nacional de Colombia
- National Science Foundation

Thank You!

Questions?