The Non-Euclidean Euclidean Algorithm and Curve Lengths on Pairs of Pants

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THE SET-UP

$G = \langle A, B \rangle$,a non-elementary subgroup of $Isom(\mathbb{H}^2)$, equivalently $PSL(2, \mathbb{R})$.

The $PSL(2, \mathbb{R})$ geometric algorithm answers the questions as to whether G discrete or not discrete.

Theorem The G-Maskit Geometric $PSL(2, \mathbb{R})$ Algorithm (1991)

Begin with A and B an ordered set of generators. There exists integers $[n_1, n_2, ..., n_t]$ such that
the sequence

$$(A, B) \to (B^{-1}, A^{-1}B^{n_1}) \to (B^{-n_1}A, B(A^{-1}B^{n_1})^{n_2}) \to \cdots \to (\tilde{A}, \tilde{B})$$

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• these numbers were never actually computed.

• interpretation of the algorithm as a non-Euclidean Euclidean Algorithm $\implies$ how to actually compute the n_i.

Notation for Algebraic and Geometric Quantities

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Hyperbolic Transformation X

Axis Ax_X

Translation length T_X

The matrix of X has a trace

$\mathrm{Tr}\ \mathrm{X}$.

$\mathrm{Tr}\ \mathrm{X}$ and T_X related by

$$\cosh \frac{T_X}{2} = \frac{1}{2} \mathrm{Tr}\ \mathrm{X}$$

Theorem

The Non-Euclidean Euclidean Algorithm (G-2013)

If one applies the Euclidean algorithm to the non-Euclidean translation lengths of the generators at each step, the output is the F-sequence $[n_1, \ldots, n_k]$.

$$n_1 = [\frac{T_A/2}{T_B/2}]$$

where [ ] *denotes the greatest integer function and*

$$n_2 = [\frac{T_B}{T_D/2}] \quad \textit{where} \quad D = A^{-1}B^{n_1}$$

and

$$n_t = [\frac{T_{A_t}/2}{T_{B_t}/2}]$$

where (A_t, B_t) *is the pair of generators at step* t *in theorem 3.*

DRAW

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There are geodesics L_A and L_B perpendicular to the axes of A and B such that

$$A = R_L \circ R_{L_A} \text{ and } B = R_L \circ H_{R_B}$$

$$A^{-1}B = R_{L_A} \circ R_{L_B}$$

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Axes are common perpendiculars:

$$AB^{-1} = R_{L_A} R_{L_B}$$

$$AB^{-2} = R_{L_A} R_{L_{B^2}}$$

$$AB^{-3} = R_{L_A} R_{L_{B^3}}$$

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Note that the length of the segment of the common perpendicular to a pair of geodesics between the two geodeiscs is the shortest of the length along any other geodesic between the two geodesics.

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Corollary

n is the n_1 of the Euclidean Algorithm.

The analogy: the Euclidean and the non-Euclidean algorithm

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- We find the sequence $[n_1, ..., n_t]$ of the discreteness algorithm by a combination of Euclidean or division algorithm type of computations with hyperbolic lengths and hyperbolic length replacements with the remainder term;
- we also at each step have an inequality rather than an equality

The Euclidean Algorithm to get the GCD:

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Begin with a and a_1

$a = n_1 a_1 + a_2$ and $0 \leq a_2 < a_1$

$$n_1 = [\frac{a}{a_1}]$$

$a_1 = n_2 a_2 + a_3$ and $0 \leq a_3 < a_2$

$$n_2 = [\frac{a_1}{a_2}]$$

$$\vdots$$

$a_{j-1} = n_j a_j + a_{j+1}$

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Begin with a and a_1

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$$n_1 = [\frac{a}{a_1}] = [\frac{T_A/2}{T_{A_1}/2}]$$

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replace with $T_{A_1}/2$ and $T_{A_2}/2$ and replace the remainder with $T_{A_3}/2$

$$n_2 = [\frac{a_1}{a_2}] = [\frac{T_{A_1}/2}{T_{A_2}/2}]$$

$$\vdots$$

$a_{j-1} = n_j a_j + a_{j+1}$ $0 \le T_{A_{j+1}}/2 \le a_{j+1} < a_j$

replace with $T_{A_{j-1}}/2$ and $T_{A_j}/2$ and the remainder with $T_{A_{j+1}}/2$

$$n_j = [\frac{a_{j-1}}{a_j}] = [\frac{T_{A_{j-1}}/2}{T_{A_j}/2}]$$

Note we have inequality rather than equality:

$$T_{A_{j+1}} \leq n_j T_{A_j} + T_{A_{j+2}}$$

Implications: New Length Inequalities

When the algorithm stops and says that the group is discrete and free, the quotient is a pair of pants and the stopping generators are two of the three unique simple closed curves on the surface. These three curves are the shortest curves on the surface.

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Three results:

1. *F-sequences* and *Translation Lengths*.
2. *Minimal curve length* and *essential self-intersections*
3. *Seam Lengths* and *Intersections*

Theorem

(G, 2015) F-sequence and translation length *If the algorithm begins with (A, B) and ends with $(\tilde{A}, \tilde{B})$, all hyperbolic.*

$$\frac{1}{2}T_A \leq \Pi_{i=1}^{t}(n_i + 1)\frac{1}{2}T_{\tilde{A}}$$

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The result follows by induction.

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often write with positive integers and simply note winding.

Primitive curves and rationals

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Theorem

(G-Keen 2002) Primitive curves correspond to rational numbers. A primitive curve will have a winding sequence $[m_1, ..., m_t]$ *for some integer* t*. Label the curve* γ_r *where* r *is the rational with continued fraction entries* $[0, m_1, ..., m_r]$

Theorem 0.3 can be rewritten as

Theorem

(Winding and Curve Lengths)

Let W be a curve corresponding to a primitive word in G that is not conjugate to a stopping generator.
Let$L(W)$ be its length and
$[n_1, ..., n_t]$ its winding sequence.
Assume that $L(S_0)$ is the longest length of any simple closed curve on the surface. Then

$$(\Pi_{i=1}^{t} n_i)L(S_0) \leq L(W) \leq (\Pi_{i=1}^{t}(n_i + 1))L(S_0).$$

Essential self-intersections for primitive curves

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Definition

If W is any closed curve on a pair of pants, it will have a certain number of essential self-intersections, that is, self-intersections along seams of the pair of pants.

All other self-intersection are *trailing intersections*.
Denote the number of essential self-intersections by

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or by

$$Int(r)$$

if $W = \gamma_r$ where γ_r is the curve with winding sequence $[m_1, ..., m_t]$ and r is the rational with continued fraction entries $[0, m_1, ..., m_t]$

Inductive formula for $Int(r)$

Theorem

(G-Keen 2002) (inductive formula for $Int(r)$)
We let $Int(r)$ denote the number of essential self-intersections of the curve with winding sequence $[n_1, ..., n_t]$, where r the corresponding rational and $r_k = [n_0, ..., n_k]$ its k-th approximent. Let $r_k = p_k/q_k$ where p_k is the numerator and q_k the denominator of the approximent and r_k is given in lowest terms.

Then the essential self-intersection numbers are given inductively as follows:

$$Int(\alpha_0) = 0, Int(\beta_0) = 0, Int(\alpha_0^{-1}\beta_0) = 0, Int(\alpha_0\beta_0) = 1, Int(\alpha_0\beta_0^2) =$$

$$Int_{p_{k+1}/q_{k+1}} =$$

$$1 + n_{k+1} \cdot Int_{p_k/q_k} + Int_{p_{k-1}/q_{k-1}}$$

Theorem

(G-2015) (Essential self-intersection number and curve length) *If $L(\gamma)$ denotes the length of a curve γ and α_0 is the shortest simple closed curve on the surface with γ_0, the longest,*

$$Int(r) \times L(\alpha_0) \leq L(\gamma_r) \leq (Int(r+1)) \times L(\gamma_0)$$

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$$Int(r) \times L(\alpha_0) \leq L(\gamma_r) \leq (Int(r+1)) \times L(\gamma_0)$$

Proof.

Basically bounded the number times the curve might go around the shortest simple curve and the longest simple curve. □

The algorithm stops at a convex right hexagon whose sides project to the three shortest curves on the surface and to the three longest seams on the pair of pants. If the alternating sides have length $x \leq y \leq z$, then the length of the seam L opposite the shortest side x satisfies:

$$\cosh L = \frac{\cosh x + \cosh y \cosh z}{\sinh y \sinh x}.$$

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Theorem (Maximal Seam Length) (G-2015) Let L_0 be the common perpendicular to the axes of A_0 and B_0 and λ_0 its image on the quotient. If ρ_i is the distance along the L_0 between its intersection with the axis of $A_0 B_0^i$ and and $A_0 B_0^{i+1}$, then

$$\lim_{i=1}^{\infty} \Sigma_{t \to \infty} \rho_i = L(L_0)$$

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Proof Look at the figure.

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