On the action of the absolute Galois group on triangle curves

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There is a well-known correspondence between

1. **Dessins d'enfants:** embedded graphs whose complement is a disjoint union of Jordan domains.
2. **Belyi pairs** (C, f)**:** coverings $f : C \to \mathbb{P}^1$ ramified over three points.
3. (Conjug. classes of) **Finite index subgroups of triangle groups** $\Delta(l, m, n) = \langle x, y, z |\ x^l = y^m = z^n = xyz = 1\rangle$

• The dessin corresponding to (C, f) is the embedded graph $f^{-1}([0, 1]) \subset C$.
• The Belyi pair corresponding to a finite index subgroup $\Gamma \subset \Delta(l, m, n)$ is $\mathbb{H}/\Gamma \to \mathbb{H}/\Delta(l, m, n)$.

Theorem *(Belyi) Belyi curves C and Belyi functions f must be defined over $\overline{\mathbb{Q}}$.*
(And so must be automorphisms of C, when $gen(C) > 1$)

Hence, there is a natural action of the **absolute Galois group**

$$Gal(\overline{\mathbb{Q}}) = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) = \{\text{field of automorphisms } \sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}\}$$

on Belyi pairs (C, f).

Example: Let $\sigma \in Gal(\overline{\mathbb{Q}})$ be such that

$$\sigma(\sqrt{2}) = -\sqrt{2} \quad \text{and} \quad \sigma(i) = -i$$

Then:

- $C : y^2 = x(x^2 - 1)(x^2 + \sqrt{2}) \Rightarrow C^\sigma : y^2 = x(x^2 - 1)(x^2 - \sqrt{2})$
- $\tau(x, y) = (-x, iy)$ on $C \Rightarrow \tau^\sigma(x, y) = (-x, -iy)$ on C^σ

A **quasiplatonic (or triangle)** curve (of type (l, m, n)) is one for which the projection

$$f : C \to C/Aut(C)$$

is a Belyi pair, i.e. $C/Aut(C)$ is an orbifold of genus 0 with three conic points (or order (l, m, n)).

The corresponding graphs

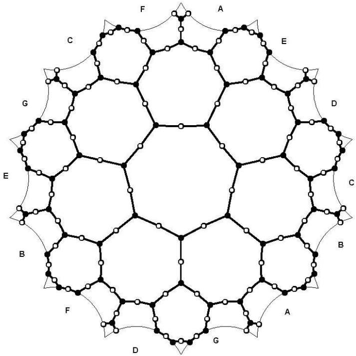
$$f^{-1}([0,1]) \subset C$$

being the so-called regular dessins (of type (l, m, n)).

Example: Klein's regular dessin of genus 3 and degree 168.

$$C : y^7 = x(x-1)^2 = 0$$
$$f(x,y) = \frac{\left(x^6+229x^5+270x^4-1695x^3+1430x^2-235x+1\right)^3\left(x^2-x+1\right)^3}{1728x(x-1)(x^3-8x^2+5x+1)^7}$$

$$\Rightarrow (C^\sigma, f^\sigma) = (C, f), \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$$



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In fact, except for the case $g = 0$ (dealt with by **Lenstra** and **Schneps**), the action is already faithful on the set of Belyi curves C of any given genus.

Theorem *(Girondo,-)*

- *The action of* $\mathrm{Gal}(\overline{\mathbb{Q}})$ *is faithful on the set of hyperelliptic curves*

$$C_{n,a} : y^2 = (x-1)(x-2)\cdots(x-2g-1)(x-a-n) \quad , a \in \overline{\mathbb{Q}}$$

(because if $\sigma(a) \neq a$, then $C_{n,a}$ is not isomorphic to $C_{n,a}^{\sigma} = C_{n,\sigma(a)}$ for many integers $n \in \mathbb{N}$.)

Questions:

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Examples of quasiplatonic curves:

- **Fermat**'s curve: $x^n + y^n = 1$
- **Klein**'s curve: $y^7 = x(x-1)^2$
- **Accola-Maclachlan**'s : $y^{2g+2} = x(x-1)(x+1)^{2g}$
- **Kulkarni**'s : $y^{2g+2} = x(x-1)^{g+2}(x+1)^{g-1}$
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• Quasiplatonic curves one usually encounters are defined over $\mathbb{Q}$. However the following theorem implies that there are such curves with coefficients as involved as one can imagine (even if we restrict ourselves to unramified Galois covers of a given one)

Theorem*(-, Jaikin-Zapirain) Let C_0 be a quasiplatonic curve of arbitrarily given hyperbolic type (l, m, n) defined over $\mathbb{Q}$. Then*

1. *$Gal(\overline{\mathbb{Q}})$ acts faithfully on the subset of quasiplatonic curves of type (l, m, n) that are unramified Galois covers of C_0.*
2. *In particular $Gal(\overline{\mathbb{Q}})$ acts faithfully on quasiplatonic curves of arbitrarily given hyperbolic type (l, m, n).*

A key tool in the proof of the theorem is the concept of the **profinite completion of a group** Γ_0. Recall that this is a group

$$\widehat{\Gamma_0} = \varprojlim \Gamma_0/\Gamma$$

that encodes the information provided by all finite quotients

$$\{\Gamma_0/\Gamma\}_{[\Gamma_0:\Gamma]<\infty}.$$

In fact, the elements of $\mathbf{x} \in \widehat{\Gamma_0}$ are written as sequences

$$\mathbf{x} = (x_\Gamma \Gamma \in \Gamma_0/\Gamma)_\Gamma \in \prod_{[\Gamma_0:\Gamma]<\infty} \Gamma_0/\Gamma$$

where $\quad x_\Gamma \equiv x_\Pi \quad (mod\ \Pi) \quad$ whenever $\Gamma \lhd \Pi$.

There are natural bijections

$$\{\textit{Finite index } \Gamma \lhd \Gamma_0\} \leftrightarrow \{\textit{Finite index } \overline{\Gamma} \lhd \widehat{\Gamma_0}\} \leftrightarrow \{\textit{Normal covers of } \mathbb{H}/\Gamma_0\}$$

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2) $Gal(\overline{\mathbb{Q}})$ does not act on $\Gamma_0 < PSL(2, \mathbb{R})$, but it does act on all finite quotients $\Gamma_0/\Gamma = Aut(\mathbb{H}/\Gamma = C \to C_0 = \mathbb{H}/\Gamma_0)$.
This gives rise to a group homomorphism

$$\begin{array}{cccc} \zeta : & Gal(\overline{\mathbb{Q}}) & \to & Out(\widehat{\Gamma_0}) = Aut(\widehat{\Gamma_0})/Inn(\widehat{\Gamma_0}) \\ & \sigma & \to & \zeta(\sigma) \end{array}$$

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3) $\zeta(\sigma)$ is compatible with the action of σ; i.e
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4) **Theorem (Hoshi-Mochizuki)** $\zeta : Gal(\overline{\mathbb{Q}}) \to Out(\widehat{\Gamma_0})$ *is injective.*

(For $\Gamma_0 = PSL(2,\mathbb{Z})$ or $\Gamma(2)$ this is a consequence of the fact that the action of $Gal(\overline{\mathbb{Q}})$ on ALL dessins is faithful).

5) **If** $\phi \in Aut(\widehat{\Delta(l,m,n)})$ satisfies $\phi(\overline{\Gamma}) = \overline{\Gamma}, \forall$ finite index $\Gamma \lhd \Delta(l,m,n)$ contained in Γ_0, **then** $\phi \in Inn(\widehat{\Delta(l,m,n)})$.

5) **If** $\phi \in Aut(\Delta\widehat{(l,m,n)})$ satisfies $\phi(\overline{\Gamma}) = \overline{\Gamma}, \forall$ finite index $\Gamma \lhd \Delta(l,m,n)$ contained in Γ_0, **then** $\phi \in Inn(\Delta\widehat{(l,m,n)})$.

6) **Let us assume** that there is $\sigma \in Gal(\overline{\mathbb{Q}})$ s.t. $C^{\sigma} \simeq C$ for all and all unramified Galois covers $C \to C_0$, **i.e let us assume** that $\Gamma^{\sigma} = \Gamma, \forall \Gamma$ as in 5). **We have to show that** $\sigma = Id$.
Now, by 3), $\zeta(\sigma)(\overline{\Gamma}) = \overline{\Gamma^{\sigma}} = \overline{\Gamma}, \forall \Gamma$; and applying 5) to $\Phi = \zeta(\sigma)$ **we deduce** that $\zeta(\sigma) \in Inn(\Delta\widehat{(l,m,n)})$, **hence** $\zeta(\sigma^d) \in Inn(\widehat{\Gamma_0})$), for $d = [\Delta(l,m,n) : \Gamma_0]$. $(\zeta(\sigma)(\widehat{\Gamma_0}) = \widehat{\Gamma_0}$ because $C_0 = \Delta/\Gamma_0$ is defined over $\mathbb{Q}$)

Finally, Hoshi-Mochizuki implies that $\sigma^d = Id.$, which, by **Artin-Schreier**'s theorem, implies that $\sigma = Id$.

7) For the second part (faithfulness on quasiplatonic curves of arbitrarily given hyperbolic type (l ,m, n))
it is enough to observe that for any type (l, m, n) there is a quasiplatonic curve C_0 of this type defined over $\mathbb{Q}$, i.e. such that $C_0^\sigma \simeq C_0$ (by **Wolfart** this is a sufficient condition: the **Earle-Shimura** phenomenon does not occur for quasiplatonic curves).
In turn, this is equivalent to proving the existence of a torsion free characteristic subgroup Γ_0 of finite index $\Gamma_0 < \Delta(l, m, n)$; for then, by 3), $\zeta(\sigma)(\overline{\Gamma_0}) = \overline{\Gamma_0}$ and, so, we could take $C_0 = \mathbb{H}/\Gamma_0$.

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$$\mathit{Take,\ \ e.g.} \qquad \Gamma_0 = \bigcap_{[\Delta(l,m,n):\Gamma]=N} \Gamma$$

Reference:

G. GD and Andrei Jaikin-Zapirain; The absolute Galois group acts faithfully on regular dessins and on Beauville surfaces.
Proc. London Math. Soc. (2015); doi: 10.1112/plms/pdv041.