Galois theory of a quaternion group origami

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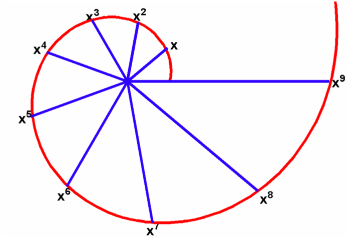
$\phi : Y \mapsto X$ $X = Y = \mathbb{G}_m =$ multiplicative group
$= \mathbb{P}^1(\mathbb{C}) - \{0, \infty\}$
$\phi : Y \to X$
$\phi : x \mapsto x^N$



Fix a rational point $p \in \mathbb{G}_m$ (thought of in the target of the map ϕ, X).

Consider the set $V = \phi^{-1}(p) = \{x \in \mathbb{G}_m \quad | \quad \phi(x) = p\}$, i.e. $\{x \in \mathbb{G}_m \quad | \quad x^N = p\}$.

$f_p(x) = x^N - p$

First, consider the case that $p = 1$. Then V is the set of (nonzero) solutions to $f_p(x) = x^N - 1$. These are the N^{th} roots of unity.

$\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times$.
$\sigma_i : \zeta_N \mapsto \zeta_N^i, (i, N) = 1$

$$\left.\begin{array}{c}\mathbb{Q}(\zeta_N)\\ \Big| \\ \mathbb{Q}\end{array}\right\} (\mathbb{Z}/N\mathbb{Z})^\times$$

When $f_p(x) = x^N - p$ is irreducible, the picture becomes the following:

$$
\begin{array}{c}
\mathbb{Q}(\sqrt[N]{p}, \zeta_N) \\
| \\
\mathbb{Q}(\zeta_N) \\
| \\
\mathbb{Q}
\end{array}
\qquad
\begin{array}{l}
\left.\vphantom{\begin{array}{c} a \\ a \end{array}}\right\} (\mathbb{Z}/N\mathbb{Z}) \\
\left.\vphantom{\begin{array}{c} a \\ a \end{array}}\right\} (\mathbb{Z}/N\mathbb{Z})^{\times}
\end{array}
$$

- $\mathrm{Gal}(sf(x^N - p)/\mathbb{Q})$ is a subgroup of $\mathrm{AGL}_1(\mathbb{Z}/N\mathbb{Z})$.

- There is a Galois representation

$$\rho_{N,p} : G_{\mathbb{Q}} \to \mathrm{AGL}_1(\mathbb{Z}/N\mathbb{Z})$$

- This is given by $\sigma \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ such that $\sigma(\zeta_N) = \zeta_N^p$ and $\frac{\sigma(\sqrt[N]{d})}{\sqrt[N]{d}} = \zeta_N^a$, so $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $b \in (\mathbb{Z}/N\mathbb{Z})$.

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Let E be an elliptic curve over $\mathbb{Q}$.

Definition

An origami is a pair (C, f) where C is a curve and $f : C \to E$ is a map branched only above one point.

We study automorphisms of origamis and relate these to polynomials over $\mathbb{Q}$.

Definition

A deck transformation or automorphism of a cover $f : C \to E$ is a homeomorphism $g : C \to C$ such that $f \circ g = f$.

Each deck transformation permutes the elements of each fiber. This defines a group action of the the deck transformations on the fibers.

Let E be an elliptic curve over $\mathbb{Q}$. Fix a positive integer N. We define multiplication by N on E, denoted $[N]$ to be adding a point to itself N times. We define the N-division points of E:

$$E[N] = \left\{ P \in E(\overline{\mathbb{Q}}) : [N]P = \mathcal{O} \right\}.$$

Facts:

- The N-division points form a group that is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$. For example, $E[2] \simeq (\mathbb{Z}/2\mathbb{Z})^2$, a Klein 4-group.
- The Galois group $G_{\mathbb{Q}}$ sends division points to division points.

We will write $\mathbb{Q}(E[N])$ to mean the field obtained by adjoining all of the coordinates of the N-division points of E to $\mathbb{Q}$.

The Galois group of $\mathbb{Q}(E[N])$ over $\mathbb{Q}$ is a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

$$\begin{array}{c} \mathbb{Q}(E[N]) \\ | \\ \mathbb{Q} \end{array} \quad \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$$

For example, the Galois group of $\mathbb{Q}(E[2])/\mathbb{Q}$ is a subgroup of the **automorphism** group of $E[2]$ and

$$\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \simeq S_3.$$

After choice of basis,

$$\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}).$$

Let E be given by $y^2 = x^3 + Ax + B$. Fix a point $P \in E(\mathbb{Q})$ given by $P = (z : w : 1)$. Consider the set

$$V = [N]^{-1}P = \left\{Q \in E(\overline{\mathbb{Q}}) | [N]Q = P\right\}.$$

For example, when $P = \mathcal{O}$, this set is the set of N-division points.

This is no longer a group in general, but we can still adjoin the coordinates of such points to $\mathbb{Q}$ and find the Galois group of the extension.

$$\begin{tikzcd}
\mathbb{Q}([N]^{-1}P) \arrow[dd, dash] & \\
 & (\mathbb{Z}/N\mathbb{Z})^2 \\
\mathbb{Q}(E[N]) \arrow[dd, dash] & \\
 & \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \\
\mathbb{Q} &
\end{tikzcd}$$

The Galois group of $\mathbb{Q}([N]^{-1}P)$ over $\mathbb{Q}$ is a subgroup of the affine general linear group

$$1 \to (\mathbb{Z}/N\mathbb{Z})^2 \to \mathrm{AGL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1$$

e.g. for $N = 2$

$$1 \to (\mathbb{Z}/2\mathbb{Z})^2 \to S_4 \to S_3 \to 1$$

$$\left\{ \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} : a, b, c, d, e, f \in (\mathbb{Z}/N\mathbb{Z}) \text{ and } ad - bc \neq 0 \right\}$$

There is a representation

$$\rho_{N,P} : G_{\mathbb{Q}} \to \mathrm{AGL}_2(\mathbb{Z}/N\mathbb{Z}).$$

- Let T_1, T_2 is a basis for $E[N]$.
- Suppose $\sigma(T_1) = aT_1 \oplus cT_2$ and $\sigma(T_2) = bT_1 \oplus dT_2$.

- Choose any $Q \in \overline{\mathbb{Q}}$ such that $[N]Q = P$.
- Suppose $\sigma(Q) \ominus Q = eT_1 \oplus fT_2$.

Then the top representation is given by

$$\sigma \mapsto \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplication by 2

$$E : y^2 = x^3 + Ax + B$$

The formula for the x-coordinate of $[2]P = P \oplus P$ $(P = (z : w : 1))$ is the following:

$$\frac{x^4 - 2Ax^2 - 8Bx + A^2}{4(x^3 + Ax + B)}$$

Proposition

Fix a rational point $P = (z : w : 1)$. Consider the extension $F_P = \mathbb{Q}(sf([2]^{-1}P))$ over $\mathbb{Q}$, where $F_P/\mathbb{Q}$ is given by the splitting field of the quartic

$$f_{E,P}(x) = (x^4 - 2Ax^2 - 8Bx + A^2) - 4z(x^3 + Ax + B).$$

If this polynomial is irreducible, then $\mathbb{Q}(sf(F_{E,P}))/\mathbb{Q}$ is an S_4-extension. Note that $S_4 = \mathrm{AGL}_2(\mathbb{Z}/2\mathbb{Z})$.

The example

$$1 \to V_4 \to S_4 \to S_3 \to 1$$

is a specific case of a more general theory. Take the semidirect product of a group and its automorphism group where the action of the quotient on the automorphism group of the normal subgroup is the identity.

$$1 \to G \to \mathrm{Hol}(G) \to \mathrm{Aut}(G) \to 1$$

Question: What about non-abelian deck groups G?
Example: Q_8 group of quaternions, non-abelian group of order 8.

An origami with the example deck group is studied in a paper titled, "An extraordinary origami curve" by Herrlich and Schithüsen.

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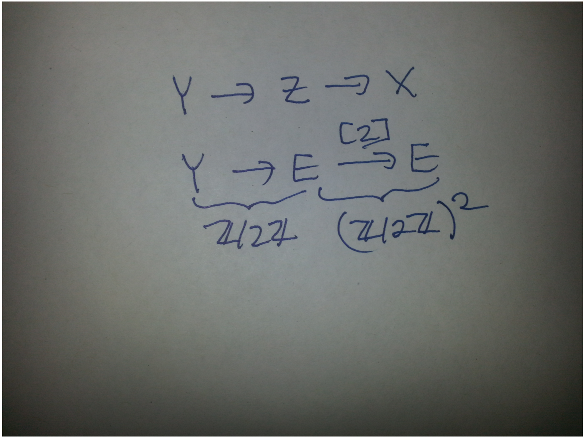
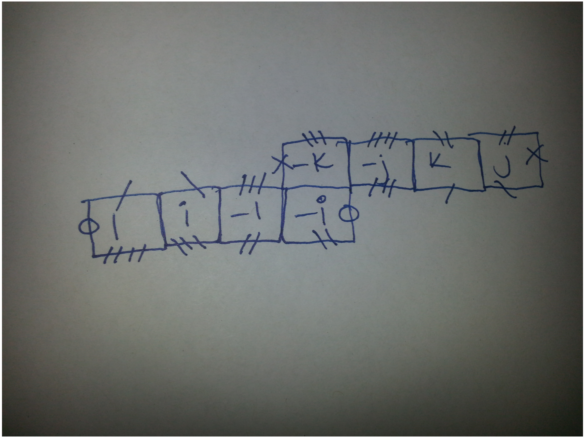

$Y \to Z \to X$
$Y \to E \xrightarrow{[2]} E$
$\mathbb{Z}/2\mathbb{Z}$
$(\mathbb{Z}/2\mathbb{Z})^2$


1
i
−1
−i
−k
−j
k
j

$$f = 8$$

$$e = \frac{8 \cdot 4}{2} = 16$$

$$v = \frac{8 \cdot 4}{4 \cdot 2} = 4$$

Formula for Euler characteristic:

$$2 - 2g = v - e + f$$

$$\implies g = 3$$

Riemann-Hurwitz formula

$$f : Y \to Z$$

Then

$$2g(Y) - 2 = \deg(f) \cdot (2g(Z) - 2) + \sum_{z \in Z} (e_z - 1)$$

Using the formula, with $g(Y) = 3$ $g(Z) = 1$, we see that there are 4 points in Y above the 2-division points in Z, each with ramification degree 2.

In fact, Herrlich and Schithüsen give that $Y : y^4 = x^3 + Ax + B$. This is an example of a superelliptic curve. The map

$$Y \to Z$$

is given by

$$(x, y) \mapsto (x, y^2)$$

.

Let g be the composition $Y \to Z \to X$. To find the $g^{-1}(P)$ points, we give a formula for multiplication by 2 in terms of the y-coordinates.

$$\phi_2 - z\psi_2^2$$

degree 4 in x

$$\omega_2 - w\psi_2^3$$

degree 6 in x (We will think of y as part of the coefficients).

The resultant polynomial for $P = (z : w : 1)$ is $y^4 - 8wy^3 + 6(2Az + 3B)y^2 - \Delta = 0$. Plugging in y^2 instead of y gives

$$f_{E,P} = y^8 - 8wy^6 + 6(2Az + 3B)y^4 - \Delta$$

Theorem (D., Goins)

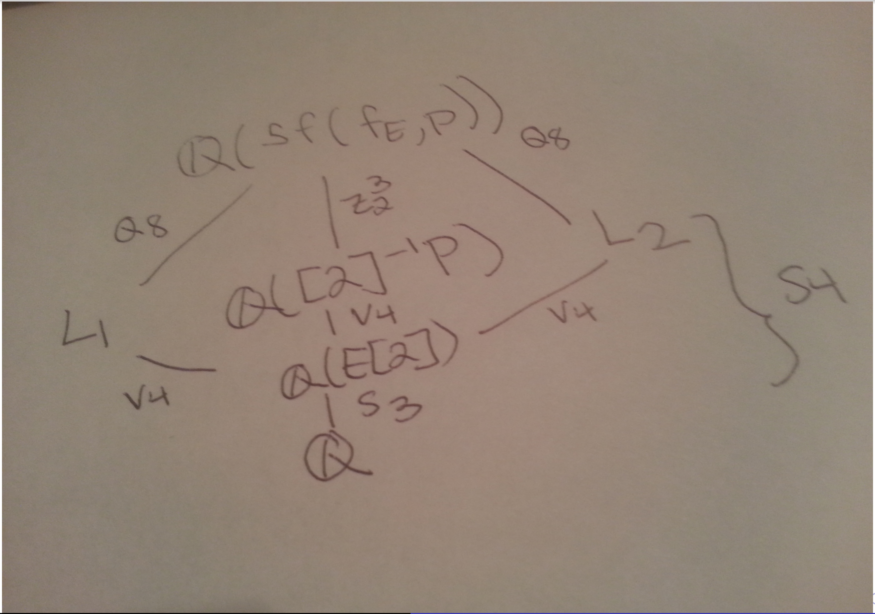
Fix a rational point $P = (z : w : 1)$. Consider the extension $F_P = \mathbb{Q}(sf(f_{E,P}))/\mathbb{Q}$ given by the splitting field of the polynomial $f_{E,P}$. If the polynomial is irreducible, then

$$\mathrm{Gal}(\mathbb{Q}(f_{E,P})/\mathbb{Q}) = \mathrm{Hol}(Q_8).$$

$$\left.\begin{array}{c} \mathbb{Q}(sf(f_{E,P})) \\ \Big| \\ \mathbb{Q} \end{array}\right\} \leq \mathrm{Hol}(Q_8)$$

$$1 \to Q_8 \to \mathrm{Hol}(Q_8) \to \mathrm{Aut}(Q_8) \simeq S_4 \to 1$$

Therefore, $\mathrm{Hol}(Q_8)$ is a specific group of order 192. Let $\Delta = -16(4A^3 + 27B^2)$.


ℚ(sf(f_E, P))
Q8
Z_2^3
Q8
L2
S4
L1
ℚ([2]^{-1}P)
V4
V4
ℚ(E[2])
V4
S3
ℚ

$$
\begin{aligned}
f_1 = {} & y^4 + 4\Delta y^3 + (512B^2\Delta - 2048Bz^3\Delta + 2048Bw^2\Delta \\
& + 2048z^6\Delta - 4096z^3w^2\Delta + 2048w^4\Delta + 6\Delta^2)y^2 \\
& + 1769472w^8\Delta^2 + 331776B^2z^6\Delta^2 - 884736B^2z^3w^2\Delta^2 + \\
& 3538944B^2w^4\Delta^2 + 512B^2\Delta^3 - 512B^2\Delta^2 - 1327104Bz^9\Delta^2 \\
& + 18432w^4\Delta^3 + 2654208Bz^6w^2\Delta^2 + 1327104Bz^3w^4\Delta^2 \\
& - 1024Bz^3\Delta^3 + 10240Bw^2\Delta^3 - 35224100536320w^6z^3\Delta^2 \\
& + 49313740750848w^4z^6\Delta^2 + 1327104z^{12}\Delta^2 \\
& - 5750784z^9w^2\Delta^2 + 9289728z^6w^4\Delta^2 + 2304z^6\Delta^3 - \\
& 6635520z^3w^6\Delta^2 - 9216z^3w^2\Delta^3 + \Delta^4
\end{aligned}
$$

Does L_1 really depend on P? There is an isomorphism from L_1 the the splitting field of $x^4 - 4\Delta x - 12A\Delta$. This is a special polynomial because it defines the S_4 extension contained inside of $\mathbb{Q}(E[4])$. What about L_2?

Thank you. Questions?