Lifting the hyperelliptic involution of a Klein surface

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joint work (in progress) with Emilio Bujalance and Peter Turbek

Let $\pi : X \to X'$ be an unbranched normal covering of compact Klein surfaces of algebraic genus bigger than one, and assume that X' is hyperelliptic.

- Let $G < \mathrm{Aut}(X)$ be the group of covering transformations: $X' = X/G$.
- Let $h_{X'} : X' \to X'$ be the hyperelliptic involution.

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Problem: Find conditions on G for $h_{X'}$ to lift to X.

This means that there exists an automorphism f in X such that $\pi f = h_{X'} \pi$.

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & X \\ \pi \downarrow & & \downarrow \pi \\ X' & \xrightarrow{\;h_{X'}\;} & X' \end{array}$$

A compact Klein surface X is the orbit space $X = S/\langle \tau \rangle$ where

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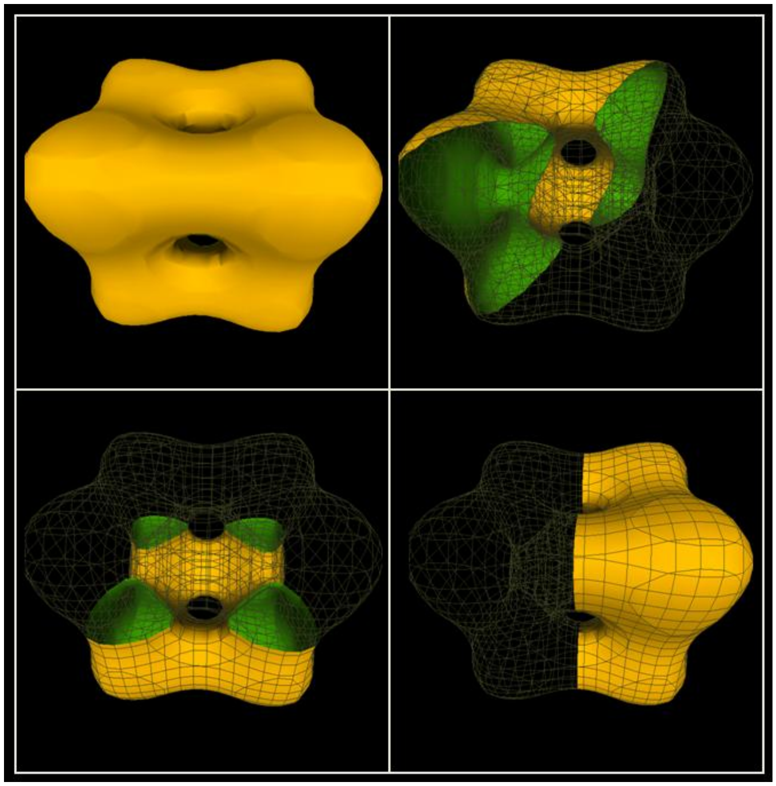

The algebraic genus of $X = S/\langle\tau\rangle$ is the genus of S.

X' is a hyperelliptic Klein surface if it admits an involution $h_{X'} : X' \to X'$ such that $X'/\langle h_{X'}\rangle$ has algebraic genus zero, that is, $X'/\langle h_{X'}\rangle$ is either the closed disc or the projective plane.

In the setting of algebraic geometry:

$$\left\{ \begin{array}{c} \text{compact Riemann} \\ \text{surfaces} \end{array} \right\} \longleftrightarrow \left\{ \text{ complex algebraic curves } \right\}$$

Theorem. (Alling & Greenleaf, 1971)

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A morphism between Klein surfaces $X \to X'$ "*differs from the corresponding concept for Riemann surfaces principally in that X may "fold" along the boundary $\partial X'$ of X'.*"

A morphism between the Klein surfaces X and X' is a continuous map $f\colon X \to X'$ such that

i) $f(\partial X) \subset \partial X'$,

ii) given $p \in X$, there exist charts $(U, z), (V, w)$ with $p \in U$ and $f(U) \subset V$ and an analytic function $F : z(U) \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccccc} U & & \xrightarrow{\quad f \quad} & & V \\ {\scriptstyle z}\downarrow & & & & \downarrow{\scriptstyle w} \\ z(U) & \xrightarrow{F} & \mathbb{C} & \xrightarrow{\Phi} & \mathbb{C}^+ \end{array}$$

where Φ is the *folding map*:

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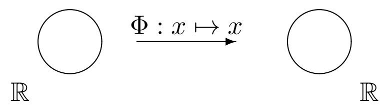

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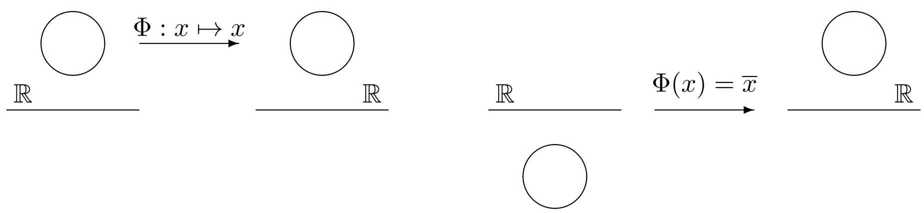

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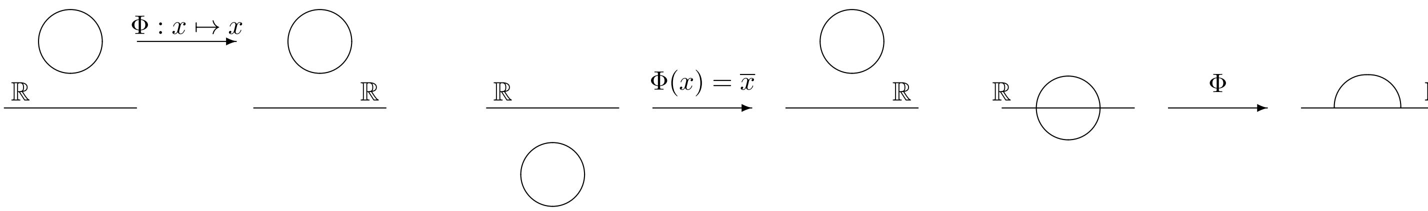

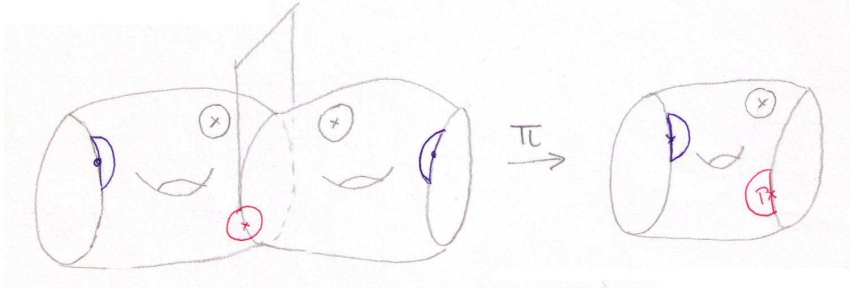
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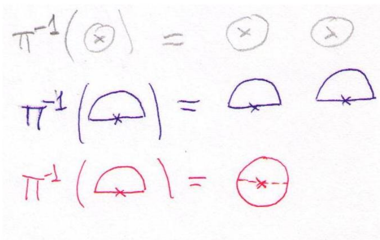
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Unbranched normal coverings $\pi : S \to S'$ of compact Riemann surfaces, S' hyperelliptic.

- Maclachlan (1971), Farkas (1976): if S is hyperelliptic and π unbranched then $G = C_2$ or $C_2 \oplus C_2$.
- Bujalance (1986): if π is unbranched and double then S is p-hyperelliptic, that is, S is a double covering of a genus p surface $\overline{S}$, for some $p \in \{0, 1, \ldots, [(g-1)/2]\}$.
- Fuertes and González-Diez (2006): equations for $\overline{S}$, which is hyperelliptic.
- Accola (1994): for each $n > 0$ there exists an n-sheeted unbranched covering where S is also hyperelliptic.
- Turbek (1997): conditions for lifting the hyperelliptic involution, unbranched case.
- Costa and Turbek (2003): conditions for lifting involutions to branched coverings.

Normal coverings $\pi : X \to X'$ of compact Klein surfaces, X' hyperelliptic.

- Bujalance, Etayo and Gamboa (1987): X is 1-hyperelliptic and π is unbranched.
- Kani (1987): classification of unbranched double coverings.

X hyperelliptic, branched coverings:

- Bujalance-C-Gamboa: topological and algebraic description
  - when X is *planar* (2007), and
  - when π is a double covering (2008).
- C-Hidalgo (2014): algebraic description of the general case.

Klein surfaces, automorphisms and NEC groups.

- An *NEC group* is a discrete subgroup $\Gamma < \mathrm{Isom}^{\pm}(\mathbb{H})$ with $\mathbb{H}/\Gamma$ compact.
- A compact Klein surface X of algebraic genus bigger than one can be written as

$$X = \mathbb{H}/\Gamma \quad \text{where } \Gamma \text{ is a } \textit{surface} \text{ NEC group,}$$

  that is, its orientation preserving elements ($\neq 1$) act fixed point free.
- A (finite) group G is a group of automorphisms of X ($X = \mathbb{H}/\Gamma$) if and only if

$$G \simeq \Gamma'/\Gamma, \quad \text{where } \Gamma' \text{ is an NEC group.}$$

Unbranched normal coverings. $\pi : X \to X'$, there exists $G < \mathrm{Aut}(X)$ such that $X' = X/G$.

Let us write $X = \mathbb{H}/\Gamma$ and $G = \Gamma'/\Gamma$ where Γ, Γ' are NEC groups and Γ is a surface NEC group. Then

$$X' = \frac{X}{G} = \frac{\mathbb{H}/\Gamma}{\Gamma'/\Gamma} = \frac{\mathbb{H}}{\Gamma'}.$$

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Unbranched: the orientation preserving elements of Γ' act fixed point free, that is, Γ' *is a surface NEC group*. Therefore Γ' uniformizes X'.

Since Γ' uniformizes X' we have $\langle h_{X'} \rangle = \Gamma'_h/\Gamma'$ for some NEC group Γ'_h. Let us write

$$\Gamma'_h = \Gamma' \cup c\,\Gamma' \quad \text{with } c \notin \Gamma'.$$

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Does c normalize Γ?

If so, then conjugation by c induces an automorphism of $\Gamma'/\Gamma = G$.

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$$\begin{CD} X @>{f}>> X \\ @V{\pi}VV @VV{\pi}V \\ X' @>{h_{X'}}>> X' \end{CD} \qquad\qquad \begin{CD} \mathbb{H}/\Gamma @>{c\,\Gamma}>> \mathbb{H}/\Gamma \\ @V{\pi}VV @VV{\pi}V \\ \mathbb{H}/\Gamma' @>{c\,\Gamma'}>> \mathbb{H}/\Gamma' \end{CD}$$

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What is the effect of conjugation by c on the generators of Γ'?

Presentations of Γ' and Γ'_h.

They depend on the *topological type* (g, k, δ) of X' :

- $g =$ topological genus;
- $k =$ number of boundary components,
- $\delta = +1$ if X' is orientable, $\delta = -1$ otherwise.

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- $g =$ topological genus;
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We divide the study in five cases (due to restrictions on the topological type of a hyperelliptic Klein surface):

- $\delta = +1$, $g = 0$, $k \geq 3$;
- $\delta = +1$, $g > 0$, $k = 1$;
- $\delta = +1$, $g > 0$, $k = 2$;
- $\delta = -1$, $g > 0$, $k > 0$;
- $\delta = -1$, $g > 0$, $k = 0$.

Example: $\boldsymbol{\delta = +1,\ g = 0,\ k \geq 3,}$ that is, $X' = \mathbb{H}/\Gamma'$ is a sphere with k holes.

$$\Gamma' \;=\; \langle\, \underbrace{c'_1, \ldots, c'_k,}_{\text{reflections}} \quad \underbrace{e'_1, \ldots, e'_k}_{\text{hyperbolic isometries}} \quad \mid \quad {c'_i}^2 = [c'_i, e'_i] = e'_1 \cdots e'_k = 1 \rangle.$$

$$\Gamma' \;\overset{2}{\triangleleft}\; \Gamma'_h = \langle\, \underbrace{c_0, \ldots, c_{2k}}_{\text{reflections}} \mid {c_i}^2 = (c_i c_{i+1})^2 = 1,\ c_0 = c_{2k} \rangle, \qquad \Gamma'_h/\Gamma' = \langle h_{X'} \rangle.$$

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We may permute cyclically the generating reflections $c_i \in \Gamma'_h$ and assume that $c_0 \notin \Gamma'$. It follows that

$$c_0, c_2, \ldots, c_{2k-2} \notin \Gamma', \quad c_1, c_3, \ldots, c_{2k-1} \in \Gamma'.$$

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So we choose

- $c'_1 = c_1,\ c'_2 = c_3,\ \ldots,\ c'_k = c_{2k-1}$ as generating reflections of Γ'.

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$$\Gamma' \;=\; \langle\, \underbrace{c_1', \ldots, c_k'}_{\text{reflections}}, \quad \underbrace{e_1', \ldots, e_k'}_{\text{hyperbolic isometries}} \quad | \quad {c_i'}^2 = [c_i', e_i'] = e_1' \cdots e_k' = 1 \rangle.$$

$$\Gamma' \;\overset{2}{\triangleleft}\; \Gamma_h' = \langle\, \underbrace{c_0, \ldots, c_{2k}}_{\text{reflections}} \mid {c_i}^2 = (c_i c_{i+1})^2 = 1,\; c_0 = c_{2k} \rangle, \qquad \Gamma_h'/\Gamma' = \langle h_{X'} \rangle.$$

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So we choose

- $c_1' = c_1,\; c_2' = c_3,\; \ldots,\; c_k' = c_{2k-1}$ as generating reflections of Γ'.

Observe that c_{2i-2} and c_{2i} commute with $c_{2i-1} = c_i'$. So we choose:

- $e_1' = c_0 c_2,\; e_2' = c_2 c_4, \ldots, e_k' = c_{2k-2} c_{2k}$ as generating hyperbolic isometries of Γ'.

  Observe that $e_1' \cdots e_i' = c_0 c_{2i}$.

Conjugation by $c_0 \in \Gamma'_h - \Gamma'$ has the following effect on the generators of Γ' :

- $c'_i \mapsto c_0 c'_i c_0 = c_0 c_{2i-1} c_0 = c_0 c_{2i}\, c_{2i-1}\, c_{2i} c_0 = (e'_1 \cdots e'_i) \cdot c'_i \cdot (e'_1 \cdots e'_i)^{-1}$.
- $e'_i \mapsto c_0 e'_i c_0 = c_0 c_{2i-2} c_{2i} c_0 = c_0 c_{2i} c_{2i}\, c_{2i-2} c_{2i} c_0 = (e'_1 \cdots e'_i) \cdot (e'_i)^{-1} \cdot (e'_1 \cdots e'_i)^{-1}$.

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Let $\mu_i = c'_i\Gamma \in \Gamma'/\Gamma = G$ and $\nu_i = e'_i\Gamma \in \Gamma'/\Gamma = G,\ \ i = 1, \ldots, k$. They generate G and satisfy

$$\mu_i^2 = [\mu_i, \nu_i] = \nu_1 \cdots \nu_k = 1.$$

If c_0 normalizes Γ then conjugation by c_0 induces this automorphism of $\Gamma'/\Gamma = G$:

$$\begin{aligned} \mu_i &\mapsto (\nu_1 \cdots \nu_i) \cdot \mu_i \cdot (\nu_1 \cdots \nu_i)^{-1}, \\ \nu_i &\mapsto (\nu_1 \cdots \nu_i) \cdot \nu_i^{-1} \cdot (\nu_1 \cdots \nu_i)^{-1}. \end{aligned}$$

Theorem. *Let $\pi : X \to X' = X/G$ be an unbranched normal covering of compact Klein surfaces where X' is hyperelliptic and topologically is a sphere with $k \geq 3$ holes. The group G can be generated by $2k$ elements $\mu_1, \dots, \mu_k$, $\nu_1, \dots, \nu_k$ which satisfy*

$$\mu_i^2 = [\mu_i, \nu_i] = \nu_1 \cdots \nu_k = 1, \quad \textit{for } i = 1, \dots, k.$$

In this situation, the hyperelliptic involution of X' lifts to X if and only if

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Corollary. *If G is abelian then the hyperelliptic involution lifts to an automorphism of X.*

Example: X' is orientable with two boundary components.

Theorem. *Let $\pi : X \to X' = X/G$ be an unbranched normal covering of compact Klein surfaces where X' is hyperelliptic and orientable with topological genus $g > 0$ and two boundary components. The group G can be generated by $2g+4$ elements $\mu_1, \ldots, \mu_g$, $\nu_1, \ldots, \nu_g$, η_1, η_2, $\varepsilon_1, \varepsilon_2$ which satisfy*

$$\eta_1^2 = \eta_2^2 = [\eta_1, \varepsilon_1] = [\eta_2, \varepsilon_2] = \varepsilon_1 \varepsilon_2 \prod_{j=1}^{g} [\mu_i, \nu_i] = 1.$$

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is a group automorphism of G, where $\pi_i = \nu_i \mu_i \nu_{i-1} \mu_{i-1} \cdots \nu_1 \mu_1$ and $P_i = [\mu_1, \nu_1] \cdots [\mu_i, \nu_i]$.

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is a group automorphism of G, where $\pi_i = \nu_i \mu_i \nu_{i-1} \mu_{i-1} \cdots \nu_1 \mu_1$ and $P_i = [\mu_1, \nu_1] \cdots [\mu_i, \nu_i]$.

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Example: X' is orientable with two boundary components.

Theorem. *Let* $\pi : X \to X' = X/G$ *be an unbranched normal covering of compact Klein surfaces where* X' *is hyperelliptic and orientable with topological genus* $g > 0$ *and two boundary components. The group* G *can be generated by* $2g+4$ *elements* $\mu_1, \ldots, \mu_g$, $\nu_1, \ldots, \nu_g$, η_1, η_2, $\varepsilon_1, \varepsilon_2$ *which satisfy*

$$\eta_1^2 = \eta_2^2 = [\eta_1, \varepsilon_1] = [\eta_2, \varepsilon_2] = \varepsilon_1 \varepsilon_2 \prod_{j=1}^{g} [\mu_i, \nu_i] = 1.$$

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Corollary. *If G is abelian and does not contain $C_2 \oplus C_2$ then the hyperelliptic involution lifts to an automorphism of X.*

With the above notations, $\eta_i = c_i'\Gamma \in \Gamma'/\Gamma = G$ for $i = 1, 2$ where c_1', c_2' are representatives of the unique conjugacy classes of elements of Γ' which fix points.

So, if $\eta_i \neq 1_G$ then the covering $\pi : X \to X/G$ folds along $\mathrm{Fix}(\eta_i)$.

Corollary. *If G is abelian and the covering is unfolded then the hyperelliptic involution lifts to an automorphism of X.*

Thank you!