Isogenous decomposition of the Jacobian of generalized Fermat curves

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Eckedal-Serre problem

1. Given $g \geq 2$, Is there a closed Riemann surface X of genus g such that JX is isogenous to the product of elliptic curves?

2. Is there a bound on the genus g with the above decomposition property?

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- R. Nakajima. On splitting of certain Jacobian varieties. J. Math. Kyoto Univ. 47 No. 2 (2007), 391-415.
- T. Shaska. Families of genus two curves with many elliptic subcovers. arXiv: 109.0434.
- T. Yamauchi. On curves with split Jacobians. Communications in Algebra 36 (2008), 1419-1425.
- T. Yamauchi. On $\mathbb{Q}$-simple factor of the Jacobian of modular curves. Yokohama J. of Math. **53** (2007), 149-160.
- J. Paulhus. Elliptic factors in Jacobians of low genus curves. Ph.D. Thesis, University of Illinois at Urbana-Champaign, (2007).

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Generalized Fermat curves

A **generalized Fermat curve of type** (p, n), with $n+1 \geq r_p$ (with $r_2 = 4$, and $r_p = 3$ for $p \geq 3$), is given by

$$S := C^p(\lambda_1, \ldots, \lambda_{n-2}) = \left\{ \begin{array}{ccc} x_1^p + x_2^p + x_3^p & = & 0 \\ \lambda_1 x_1^p + x_2^p + x_4^p & = & 0 \\ \lambda_2 x_1^p + x_2^p + x_5^p & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{n-2} x_1^p + x_2^p + x_{n+1}^p & = & 0 \end{array} \right\} \subset \mathbb{P}^n,$$

where $\lambda_1, \ldots, \lambda_{n-2} \in \mathbb{C} - \{0, 1\}$ and, for $i \neq j$, $\lambda_i \neq \lambda_j$.

Properties of S

The Riemann surface S has genus $g = 1 + \frac{\phi(p,n)}{2} \geq 1$ and it admits, as a group of conformal automorphisms, the generalized Fermat group $H_0 \cong \mathbb{Z}_p^n$; generated by the transformations

$$a_j([x_1 : \cdots : x_{n+1}]) = [x_1 : \cdots : x_{j-1} : \omega_p x_j : x_{j+1} : \cdots : x_{n+1}], \ ,$$

with $j = 1, \ldots, n$, where $\omega_p = e^{2\pi i/p}$.

We set $a_{n+1} = a_1^{-1} \cdots a_n^{-1}$, that is,

$$a_{n+1}([x_1 : \cdots : x_{n+1}]) = [x_1 : \cdots : x_n : \omega_p x_{n+1}].$$

Decomposition of the Jacobian of Generalized Fermat curves

Theorem

Let (S, H) be a generalized Fermat pair of type (p, n), where p is a prime integer. Then

$$JS \cong_{isog.} \prod_{H_r} JS_{H_r},$$

where H_r runs over all subgroups of H_0 which are isomorphic to $\mathbb{Z}_p^{n-1}$ and such that S/H_r has genus at least one, and S_{H_r} is the underlying Riemann surface of the orbifold S/H_r.

The cyclic p-gonal curves S_{H_r} runs over all curves of the form

$$y^p = \prod_{j=1}^{r} (x - \mu_j)^{\alpha_j},$$

where $\{\mu_1, \ldots, \mu_r\} \subset \{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}\}$, $\mu_i \neq \mu_j$ if $i \neq j$, $\alpha_j \in \{1, 2, \ldots, p-1\}$ satisfying the following.

(i) If every $\mu_j \neq \infty$, then $\alpha_1 = 1$, $\alpha_2 + \cdots + \alpha_r \equiv p - 1 \mod (p)$;

(ii) If some $\lambda_a = \infty$, then $\alpha_1 + \cdots + \alpha_{a-1} + \alpha_{a+1} + \cdots + \alpha_r \equiv p - 1 \mod (p)$.

Kani-Rosen docomposition theorem (1989)

Theorem

Let S be a closed Riemann surface of genus $g \geq 1$ and let $H_1, \dots, H_r < \mathrm{Aut}(S)$ such that:

1. $H_i H_j = H_j H_i$, *for all* $i, j = 1, \dots, r$;
2. *there are integers* $n_1, \dots, n_r$ *satisfying that*
   i. $\sum_{i,j=1}^{r} n_i n_j g_{H_i H_j} = 0$, *and*
   ii. *for every* $i = 1, \dots, r$, *it also holds that* $\sum_{j=1}^{r} n_j g_{H_i H_j} = 0$.

Then

$$\prod_{n_i > 0} (JS_{H_i})^{n_i} \cong_{isog.} \prod_{n_j < 0} \left(JS_{H_j}\right)^{-n_j}.$$

Corollary

Let S be a closed Riemann surface of genus $g \geq 1$ and let $H_1, \ldots, H_s < \mathrm{Aut}(S)$ such that:

1. $H_i H_j = H_j H_i$, for all $i, j = 1, \ldots, s$;
2. $g_{H_i H_j} = 0$, for $1 \leq i < j \leq s$
3. $g = \sum_{j=1}^{s} g_{H_j}$.

Then

$$JS \cong_{isog.} \prod_{j=1}^{s} JS_{H_j}.$$

Counting formula

Lemma

Let $q \geq 2$ and $r \geq 2$ be integers and let $\psi_q(r)$ be the number of different tuples $(\alpha_2, \dots, \alpha_r)$ so that $\alpha_j \in \{1, 2, \dots, q-1\}$, and $\alpha_2 + \cdots + \alpha_r \equiv -1 \mod (q)$. Then

$$\psi_q(r) = (-1)^{r+1} \left(\frac{(1-q)^{r-1} - 1}{q} \right).$$

proof

Let us consider a tuple $(\alpha_2, \ldots, \alpha_{r-1}, \alpha_r)$, where $\alpha_j \in \{1, \ldots, q-1\}$ and $\alpha_2 + \cdots + \alpha_r \equiv -1 \mod (q)$. Since α_r is not congruent to 0 mod q, we must have that $\alpha_2 + \cdots + \alpha_{r-1}$ cannot be congruent to -1 mod q. But this last sum can be congruent to any value inside $\{0, 1, \ldots, q-2\}$. We also note that α_r gets uniquely determined by $\alpha_1, \ldots, \alpha_{r-1}$. In this way,

$$\psi_q(r) = (q-1)^{r-2} - \psi_q(r-1)$$

This recurrence asserts that

$$\begin{aligned}\psi_q(r) &= \sum_{k=2}^{r}(-1)^k(q-1)^{r-k} \\ &= (-1)^r\sum_{k=2}^{r}(1-q)^{r-k} \\ &= (-1)^r\sum_{k=0}^{r-2}(1-q)^{k} \\ &= (-1)^{r+1}\left(\frac{(1-q)^{r-1}-1}{q}\right)\end{aligned}$$

We will need the following equality, to write the genus of a generalized Fermat curve as the sum of the genus of cyclic gonal curves (we will use this for the prime case).

Lemma

Let $n, q \geq 2$ be integers with $n+1 \geq r_q$, where $r_2 = 4$ and $r_q = 3$ for $q \geq 3$. Then

$$1 + \frac{\phi(q,n)}{2} = \sum_{r=r_q}^{n+1} \binom{n+1}{r} \frac{(r-2)(q-1)}{2} \psi_q(r).$$