Isogenous decomposition of the Jacobian of generalized Fermat curves

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Eckedal-Serre problem

1. Given $g \geq 2$, Is there a closed Riemann surface $X$ of genus $g$ such that $JX$ is isogenous to the product of elliptic curves?

2. Is there a bound on the genus $g$ with the above decomposition property?
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A generalized Fermat curve of type \((p, n)\), with \(n + 1 \geq r_p\) (with \(r_2 = 4\), and \(r_p = 3\) for \(p \geq 3\)), is given by

\[
S := C^p(\lambda_1, \ldots, \lambda_{n-2}) = \begin{cases}
  x_1^p + x_2^p + x_3^p &= 0 \\
  \lambda_1 x_1^p + x_2^p + x_4^p &= 0 \\
  \lambda_2 x_1^p + x_2^p + x_5^p &= 0 \\
  \vdots & \vdots & \vdots \\
  \lambda_{n-2} x_1^p + x_2^p + x_{n+1}^p &= 0 
\end{cases} \subset \mathbb{P}^n,
\]

where \(\lambda_1, \ldots, \lambda_{n-2} \in \mathbb{C} - \{0, 1\}\) and, for \(i \neq j\), \(\lambda_i \neq \lambda_j\).
Properties of $S$

The Riemann surface $S$ has genus $g = 1 + \frac{\phi(p,n)}{2} \geq 1$ and it admits, as a group of conformal automorphisms, the generalized Fermat group $H_0 \cong \mathbb{Z}_p^n$; generated by the transformations

$$a_j([x_1 : \cdots : x_{n+1}]) = [x_1 : \cdots : x_{j-1} : \omega_p x_j : x_{j+1} : \cdots : x_{n+1}],$$

with $j = 1, \ldots, n$, where $\omega_p = e^{2\pi i/p}$.

We set $a_{n+1} = a_1^{-1} \cdots a_n^{-1}$, that is,

$$a_{n+1}([x_1 : \cdots : x_{n+1}]) = [x_1 : \cdots : x_n : \omega_p x_{n+1}].$$
Deformation of the Jacobian of Generalized Fermat curves

**Theorem**

Let \((S, H)\) be a generalized Fermat pair of type \((p, n)\), where \(p\) is a prime integer. Then

\[ JS \cong \bigoplus_{H_r} JS_{H_r}, \]

where \(H_r\) runs over all subgroups of \(H_0\) which are isomorphic to \(\mathbb{Z}_{p}^{n-1}\) and such that \(S/H_r\) has genus at least one, and \(S_{H_r}\) is the underlying Riemann surface of the orbifold \(S/H_r\).
The cyclic $p$-gonal curves $S_{H_r}$ runs over all curves of the form

$$y^p = \prod_{j=1}^{r} (x - \mu_j)^{\alpha_j},$$

where $\{\mu_1, \ldots, \mu_r\} \subset \{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}\}$, $\mu_i \neq \mu_j$ if $i \neq j$, $\alpha_j \in \{1, 2, \ldots, p-1\}$ satisfying the following.

(i) If every $\mu_j \neq \infty$, then $\alpha_1 = 1$, $\alpha_2 + \cdots + \alpha_r \equiv p - 1 \mod (p)$;

(ii) If some $\lambda_a = \infty$, then

$$\alpha_1 + \cdots + \alpha_{a-1} + \alpha_{a+1} + \cdots + \alpha_r \equiv p - 1 \mod (p).$$

**Theorem**

Let $S$ be a closed Riemann surface of genus $g \geq 1$ and let $H_1, \ldots, H_r < \text{Aut}(S)$ such that:

1. $H_i H_j = H_j H_i$, for all $i, j = 1, \ldots, r$;
2. there are integers $n_1, \ldots, n_r$ satisfying that
   i. $\sum_{i,j=1}^{r} n_i n_j g_{H_i H_j} = 0$, and
   ii. for every $i = 1, \ldots, r$, it also holds that $\sum_{j=1}^{r} n_j g_{H_i H_j} = 0$.

Then

$$\prod_{n_i > 0} (JS_{H_i})^{n_i} \cong_{\text{isog.}} \prod_{n_j < 0} (JS_{H_j})^{-n_j}.$$
Corollary

Let $S$ be a closed Riemann surface of genus $g \geq 1$ and let $H_1, \ldots, H_s < \text{Aut}(S)$ such that:

1. $H_i H_j = H_j H_i$, for all $i, j = 1, \ldots, s$;
2. $g_{H_i H_j} = 0$, for $1 \leq i < j \leq s$
3. $g = \sum_{j=1}^{s} g_{H_j}$.

Then

$$JS \cong_{\text{isog.}} \prod_{j=1}^{s} JS_{H_j}.$$
Counting formula

Lemma

Let \( q \geq 2 \) and \( r \geq 2 \) be integers and let \( \psi_q(r) \) be the number of different tuples \((\alpha_2, \ldots, \alpha_r)\) so that \( \alpha_j \in \{1, 2, \ldots, q - 1\} \), and \( \alpha_2 + \cdots + \alpha_r \equiv -1 \mod (q) \). Then

\[
\psi_q(r) = (-1)^{r+1} \left( \frac{(1 - q)^{r-1} - 1}{q} \right).
\]
Let us consider a tuple \((\alpha_2, \ldots, \alpha_{r-1}, \alpha_r)\), where \(\alpha_j \in \{1, \ldots, q - 1\}\) and \(\alpha_2 + \cdots + \alpha_r \equiv -1 \mod (q)\). Since \(\alpha_r\) is not congruent to 0 mod \(q\), we must have that \(\alpha_2 + \cdots + \alpha_{r-1}\) cannot be congruent to \(-1\) mod \(q\). But this last sum can be congruent to any value inside \(\{0, 1, \ldots, q - 2\}\). We also note that \(\alpha_r\) gets uniquely determined by \(\alpha_1, \ldots, \alpha_{r-1}\). In this way,

\[
\psi_q(r) = (q - 1)^{r-2} - \psi_q(r - 1)
\]
This recurrence asserts that

\[ \psi_q(r) = \sum_{k=2}^{r} (-1)^k (q - 1)^{r-k} \]

\[ = (-1)^r \sum_{k=2}^{r} (1 - q)^{r-k} \]

\[ = (-1)^r \sum_{k=0}^{r-2} (1 - q)^k \]

\[ = (-1)^{r+1} \left( \frac{(1 - q)^{r-1} - 1}{q} \right) \]
We will need the following equality, to write the genus of a generalized Fermat curve as the sum of the genus of cyclic gonal curves (we will use this for the prime case).

**Lemma**

Let \( n, q \geq 2 \) be integers with \( n + 1 \geq r_q \), where \( r_2 = 4 \) and \( r_q = 3 \) for \( q \geq 3 \). Then

\[
1 + \frac{\phi(q, n)}{2} = \sum_{r=r_q}^{n+1} \binom{n+1}{r} \frac{(r-2)(q-1)}{2} \psi_q(r).
\]