Symmetric surfaces with quasi-platonic $\text{PSL}(2,q)$ action
Preliminary report

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Overview of sections

- Symmetric Quasi-Platonic (QP) Actions.
- Mirrors of Symmetries.
- Symmetric QP $PSL_2(q)$ actions - prior and new results (Macbeath, Singerman, Broughton et. al, Tyszkowska).

Motivations

- Why symmetries? Symmetries are complex conjugations of a surface defined over $\mathbb{R}$. The mirrors are real curves.
- Why $PSL_2(q)$? simple group, many low genus actions, numerous symmetries, nice group structure, easy calculations.
QP actions and symmetric QP actions

QP actions - definition

Definition

The finite group $G$ acts conformally on the closed, orientable Riemann surface $S$ if there is a monomorphism:

$$\epsilon : G \rightarrow \text{Aut}(S),$$

where $\text{Aut}(S)$ is the conformal automorphism group of $S$.

An action is quasi-platonic if:

- The quotient surface has genus zero: $S/G \cong P^1(\mathbb{C})$.
- The quotient map: $\pi_G : S \rightarrow S/G = P^1(\mathbb{C})$ is ramified over three points.
Symmetric QP actions - example

- Discuss: QP-action in this picture.
- Explain the term quasi-platonic.
- Show symmetries, mirrors, and separation.
Overview
Symmetric QP-actions
Mirrors
Symmetric $PSL(2, q)$ actions

QP actions and symmetric QP actions

**Construction of actions - summary**

- Let $p, q, r$ be the reflections in the sides of a (hyperbolic) triangle on the surface with angles $\frac{2\pi}{l}, \frac{2\pi}{m}, \frac{2\pi}{n}$.
- Let $a = pq, b = qr, c = rp$ be the rotations at the corners of the triangle. Define an action by identifying a triple $(a, b, c) \in G^3$ with the rotations of the same name.
- We call $(a, b, c)$ a generating $(l, m, n)$-triple of $G$. The assignment defines an action as long as the following hold.

  
  $G = \langle a, b, c \rangle$ \hspace{1cm} (1)

  $o(a) = l, \ o(b) = m, \ o(c) = n$ \hspace{1cm} (2)

  $abc = 1$ \hspace{1cm} (3)

- The genus $\sigma$ of $S$ satisfies

  
  $2\sigma - 2 = |G| \left( 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n} \right)$
Symmetries and QP actions

Definition

A symmetry or reflection on a surface $S$ is an anti-conformal involution $\varphi$ of $S$. A QP action $\epsilon : G \to \text{Aut}(S)$ is symmetric if there is a symmetry $\varphi$ normalizing the action of $G$, namely $\varphi\epsilon(G)\varphi = \epsilon(G)$.

For a symmetric QP action:

- Denote by $\theta$ the induced involutary automorphism of $G$

  $$\theta(g) = \epsilon^{-1}(\varphi\epsilon(g)\varphi).$$

- Define $G^* = \langle \theta \rangle \rtimes G$, and extend the action $\epsilon : G^* \to \text{Aut}^*(S) = \langle \text{Aut}(S), \varphi \rangle$ by $\theta \to \varphi$. 
Macbeath-Singerman symmetries

- If $\varphi = q$ is a symmetry then automorphism $\theta$ satisfies
  \[ \theta(a) = a^{-1}, \quad \theta(b) = b^{-1}. \]
  with similar formulas for $p$ and $q$.
- The reflections $p, q, r$ are sometimes called Macbeath-Singerman symmetries.
- The local reflection $q$ extends to symmetry of the entire surface if and only if an automorphism $\theta$, satisfying the above equation, exists.
Symmetries come in two types, depending on whether $\theta$ is an inner or outer automorphism.

- If $\theta$ is inner, then $G^* \simeq G \times \mathbb{Z}_2$ assuming $G$ is centerless. In this case $S$ has fixed point free symmetries.

- If $\theta$ is outer then $S$ has no fixed point free symmetries. If $G = PSL_2(q)$ then $G^* = PGL_2(q)$
Mirrors - definitions

Definition

Let $\varphi$ be a symmetry or reflection on a surface $S$.

- The fixed point set $\mathcal{M}_{\varphi}$ of $\varphi$ is called the mirror of the symmetry.
- The mirror $\mathcal{M}_{\varphi}$ is a disjoint union of circles called ovals.
- The symmetry $\varphi$ is called separating if $S - \mathcal{M}_{\varphi}$ consists of two disjoint mirror image pieces, otherwise it is called non-separating.
Number of ovals

- The centralizer
  \[ \text{Cent}_G(q) = \{ g \in G : \theta(g) = g \} \]
  acts transitively on the ovals of the mirror of \( q \).
- The stabilizer of an oval is dihedral or cyclic (next slide).
- The subgroup of rotations of the stabilizer of an oval can be computed from the edge pattern of the oval (next slide).
- Similar results hold for \( p \) and \( r \).
- The number of ovals can be computed using the orbit-stabilizer theorem.
Edge patterns and oval rotations

- Discuss: Show PQR patterns of ovals.
- Describe oval rotation.
### Partial edge pattern table

- **Four of eight patterns shown.**
- \( l = 2\lambda + 1 \) or \( l = 2\lambda \)
  - \( m = 2\mu + 1 \) or \( m = 2\mu \)
  - \( n = 2\nu + 1 \) or \( n = 2\nu \)

<table>
<thead>
<tr>
<th>even exponents</th>
<th>odd exponents</th>
<th>basic PQR pattern</th>
<th>oval rotation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l, m, n )</td>
<td>( l, m, n )</td>
<td>( P^+Q^+R^+ )</td>
<td>( a^{-\lambda}b^{-\mu}c^{-\nu} )</td>
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<tr>
<td>( l )</td>
<td>( m, n )</td>
<td>( Q^+R^+P^+P^-R^-Q^- )</td>
<td>( b^{-\mu}c^{-\nu}a^{-\lambda}c^\nu b^\mu a^\lambda )</td>
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<tr>
<td>( l, m )</td>
<td>( n )</td>
<td>( Q^+Q^-R^+P^+P^-R^- )</td>
<td>( b^\mu a^\lambda )</td>
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<td>( l, m, n )</td>
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<td>( P^+P^-Q^+Q^-R^+R^- )</td>
<td>( c^{-\nu}a^\lambda c^\nu b^\mu )</td>
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<td>( P^+P^- )</td>
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<td></td>
<td>( R^+R^- )</td>
<td>( c^\nu b^\mu )</td>
</tr>
</tbody>
</table>
We have two ways of determining separability of mirrors. One way uses fixed point formulas. We first note a fixed point formula for \( g \in G \).

**Proposition**

*Fixed point formula for \( g \in G \)*

\[
|S^g| = |N_G(\langle g \rangle)| (\delta_l(g)/l + \delta_m(g)/m + \delta_n(g)/n)
\]

(4)

where \( \delta_l(g) = 1 \) if \( g \) is conjugate to a power of \( a \), and \( 0 \) otherwise. Similar definitions for \( \delta_m(g) \), \( \delta_n(g) \).
Mirrors - separability - 1

For each pattern $B_i$ let $\mathcal{M}(\varphi, B_i)$ be the union of all the ovals in $\mathcal{M}_\varphi$ with edge pattern $B_i$.

**Theorem**

Suppose $a$ has even order and let $h$ be the involution $a^\lambda$. Then if $q$ is separating we have the fixed point inequality:

$$|S^h| \leq 2 |\mathcal{M}(q, B_1)| + \cdots + 2 |\mathcal{M}(q, B_s)|.$$  

**Theorem**

Suppose $a$ has odd order. Then if $q$ is separating we have the fixed point equality:

$$\frac{|N_G(a)|}{1} = |N_G(a) \cap \text{Cent}_G(q)|$$
Notes:

- If $q$ is separating then all the fixed points of $a$ or $h$ must lie on the mirror $M_q$. The right hand side of the equations count or estimate the number of these fixed points.

- Typically one proves that symmetries are non-separating by showing that the left hand sides are much larger than the right hand sides.

- The fixed point formulas don’t always work.
The second method involves counting triangles.

**Theorem**

There is an easily implemented computer algorithm that counts all the elements of $G^*$ corresponding to triangles lying on one side of the mirror $M_q$. The mirror $M_q$ is separating if and only if the count terminates with $|G| = |G^*|/2$ elements.
Some results and conjectures

Prior results

**Theorem**

*(Macbeath)* Every quasi-platonic action of $\text{PSL}_2(q)$ has Macbeath-Singerman symmetries.

- *(Broughton et al.)* The symmetries of Hurwitz actions have been completely characterized. All symmetries with non-empty mirrors symmetries are non-separating. Trace arguments in $\text{SL}(2, q)$ are used.

- *(Tyszkowska)* There is a Harnack like theorem describing the maximum number of ovals of a Macbeath-Singerman symmetry of a $\text{PSL}(2, q)$ action. The oval rotations and trace arguments are used.
Some results and conjectures

Easy results

- (originally proven in Broughton et al. for Hurwitz actions) The Macbeath Singerman symmetries are all conjugate. Therefore every mirror has ovals of each pattern type.
- If $q$ is not divisible 3 and $q > 7$ then all of the Macbeath-Singerman symmetries for hyperbolic $(2, 3, n)$ and $(3, 3, n)$ actions are non-separating.
Conjecture

All Macbeath-Singerman symmetries of $\text{PSL}_2(q)$ are non-separating.

See next slide.
Sample results

Show tables of Symmetric QP-actions
Computational Tools

We used Magma to compute all the results shown. For faster computation and for general results we can try traces in $SL_2(q)$ as follows (see Broughton et al, Glover-Sjerve and Tyszkowska).

- For each possible order $n$ of an element in $u \in PSL_2(q)$ there is a universal polynomial $f_n(\tau) \in \mathbb{Z}[\tau]$ such that $ord(u) = n$ if and only if $f_n(Tr(U) = 0$ for any covering element of $u$.
- The trace of any word in $A, B, C$ can be easily computed as a polynomial in $\{\alpha, \beta, \gamma\}$. In particular this can be done for oval rotations.
Any Questions?
References


References - continued

