Symmetric surfaces with quasi-platonic PSL(2,q) action

Preliminary report

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Overview

Overview of sections

- Symmetric Quasi-Platonic (QP) Actions.
- Mirrors of Symmetries.
- Symmetric QP $PSL_2(q)$ actions - prior and new results (Macbeath, Singerman, Broughton et. al, Tyszkowska).

Motivations

- Why symmetries? Symmetries are complex conjugations of a surface defined over $\mathbb{R}$. The mirrors are real curves.
- Why $PSL_2(q)$?
  simple group, many low genus actions, numerous symmetries, nice group structure, easy calculations.

QP actions - definition

Definition

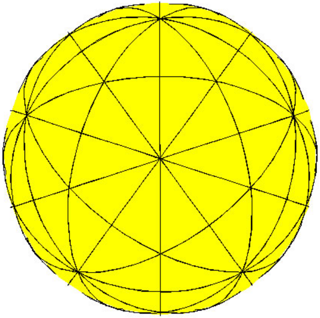
- The finite group G acts conformally on the closed, orientable Riemann surface S if there is a monomorphism:

$$\epsilon : G \to \mathrm{Aut}(S),$$

  where $\mathrm{Aut}(S)$ is the conformal automorphism group of S.
- An action is quasi-platonic if:
  - The quotient surface has genus zero: $S/G \simeq P^1(\mathbb{C})$.
  - The quotient map: $\pi_G : S \to S/G = P^1(\mathbb{C})$ is ramified over three points.

Symmetric QP actions - example

- Discuss: QP-action in this picture.
- Explain the term quasi-platonic.
- Show symmetries, mirrors, and separation.



Construction of actions - summary

- Let p, q, r be the reflections in the sides of a (hyperbolic) triangle on the surface with angles $\frac{2\pi}{l}, \frac{2\pi}{m}, \frac{2\pi}{n}$.
- Let $a = pq$, $b = qr$, $c = rp$ be the rotations at the corners of the triangle. Define an action by identifying a triple $(a, b, c) \in G^3$ with the rotations of the same name.
- We call (a, b, c) a generating (l, m, n)-triple of G. The assignment defines an action as long as the following hold.

$$G = \langle a, b, c \rangle \tag{1}$$

$$o(a) = l,\ o(b) = m,\ o(c) = n \tag{2}$$

$$abc = 1 \tag{3}$$

- The genus σ of S satisfies

$$2\sigma - 2 = |G| \left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right)$$

Symmetries and QP actions

Definition

A *symmetry* or *reflection* on a surface S is an anti-conformal involution φ of S. A QP action $\epsilon : G \to \mathrm{Aut}(S)$ is *symmetric* if there is a symmetry φ normalizing the action of G, namely $\varphi\epsilon(G)\varphi = \epsilon(G)$.

For a symmetric QP action:

- Denote by θ the induced involutary automorphism of G

$$\theta(g) = \epsilon^{-1}(\varphi\epsilon(g)\varphi).$$

- Define $G^* = \langle\theta\rangle \ltimes G$, and extend the action $\epsilon : G^* \to \mathrm{Aut}^*(S) = \langle \mathrm{Aut}(S), \varphi\rangle$ by $\theta \to \varphi$.

Macbeath-Singerman symmetries

- If $\varphi = q$ is a symmetry then automorphism θ satisfies

$$\theta(a) = a^{-1},\ \theta(b) = b^{-1}.$$

  with similar formulas for p and q.
- The reflections p, q, r are sometimes called Macbeath - Singerman symmetries.
- The local reflection q extends to symmetry of the entire surface if and only if an automorphism θ, satisfying the above equation, exists.

Types of symmetries

Symmetries come in two types, depending on whether θ is an inner or outer automorphism.

- If θ is inner, then $G^* \simeq G \times \mathbb{Z}_2$ assuming G is centerless. In this case S has fixed point free symmetries.
- If θ is outer then S has no fixed point free symmetries. If $G = PSL_2(q)$ then $G^* = PGL_2(q)$

Mirrors - definitions

Definition

Let φ be a *symmetry* or *reflection* on a surface S.

- The fixed point set $\mathcal{M}_\varphi$ of φ is called the mirror of the symmetry.
- The mirror $\mathcal{M}_\varphi$ is a disjoint union of circles called *ovals*.
- The symmetry φ is called separating if $S - \mathcal{M}_\varphi$ consists of two disjoint mirror image pieces, otherwise it is called non-separating.

Number of ovals

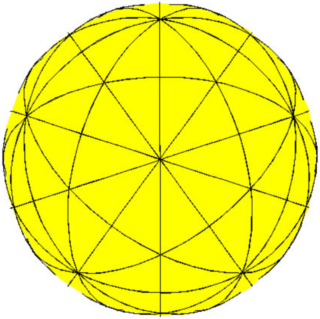
- The centralizer

$$\mathrm{Cent}_G(q) = \{g \in G : \theta(g) = g\}$$

  acts transitively on the ovals of the mirror of q.
- The stabilizer of an oval is dihedral or cyclic (next slide).
- The subgroup of rotations of the stabilizer of an oval can be computed from the edge pattern of the oval (next slide).
- Similar results hold for p and r.
- The number of ovals can be computed using the orbit-stabilizer theorem.

Edge patterns and oval rotations

- Discuss: Show PQR patterns of ovals.
- Describe oval rotation.



Partial edge pattern table

- Four of eight patterns shown.
- $l = 2\lambda + 1$ or $l = 2\lambda$
  $m = 2\mu + 1$ or $m = 2\mu$
  $n = 2\nu + 1$ or $n = 2\nu$

even exponents	odd exponents	basic PQR pattern	oval rotation
	l, m, n	$P^+Q^+R^+$	$a^{-\lambda}b^{-\mu}c^{-\nu}$
l	m, n	$Q^+R^+P^+P^-R^-Q^-$	$b^{-\mu}c^{-\nu}a^{-\lambda}c^{\nu}b^{\mu}a^{\lambda}$
l, m	n	Q^+Q^- $R^+P^+P^-R^-$	$b^{\mu}a^{\lambda}$ $c^{-\nu}a^{\lambda}c^{\nu}b^{\mu}$
l, m, n		P^+P^- Q^+Q^- R^+R^-	$a^{\lambda}c^{\nu}$ $b^{\mu}a^{\lambda}$ $c^{\nu}b^{\mu}$

Fixed point formula

We have two ways of determining separability of mirrors. One way uses fixed point formulas. We first note a fixed point formula for $g \in G$.

Proposition

Fixed point formula for $g \in G$

$$|S^g| = |N_G(\langle g \rangle)|(\delta_l(g)/l + \delta_m(g)/m + \delta_n(g)/n) \qquad (4)$$

where $\delta_l(g) = 1$ if g is conjugate to a power of a, and 0 otherwise. Similar definitions for $\delta_m(g)$, $\delta_n(g)$.

Mirrors - separability - 1

For each pattern B_i let $\mathcal{M}(\varphi, B_i)$ be the union of all the ovals in $\mathcal{M}_\varphi$ with edge pattern B_i.

Theorem

Suppose a has even order and let h be the involution a^λ. Then if q is separating we have the fixed point inequality:

$$\left|S^h\right| \leq 2\left|\mathcal{M}(q, B_1)\right| + \cdots + 2\left|\mathcal{M}(q, B_s)\right|.$$

Theorem

Suppose a has odd order. Then if q is separating we have the fixed point equality:

$$\frac{|N_G(a)|}{l} = |N_G(a) \cap \mathrm{Cent}_G(q)|$$

Mirrors - separability - 2

Notes:

- If q is separating then all the fixed points of a or h must lie on the mirror $\mathcal{M}_q$. The right hand side of the equations count or estimate the number of these fixed points.
- Typically one proves that symmetries are non-separating by showing that the left hand sides are much larger than the right hand sides.
- The fixed point formulas don't always work.

Mirrors - separability - 3

The second method involves counting triangles.

Theorem

There is an easily implemented computer algorithm that counts all the elements of G^* corresponding to triangles lying on one side of the mirror $\mathcal{M}_q$. The mirror $\mathcal{M}_q$ is separating if and only if the count terminates with $|G| = |G^*|/2$ elements.

Prior results

Theorem

(Macbeath) Every quasi-platonic action of $PSL_2(q)$ has Macbeath-Singerman symmetries.

- (Broughton et al.) The symmetries of Hurwitz actions have been completely characterized. All symmetries with non-empty mirrors symmetries are non-separating. Trace arguments in $SL(2, q)$ are used.
- (Tyszkowska)There is a Harnack like theorem describing the maximum number of ovals of a Macbeath-Singerman symmetry of a $PSL(2, q)$ action. The oval rotations and trace arguments are used.

Easy results

- (originally proven in Broughton et al. for Hurwitz actions) The Macbeath Singerman symmetries are all conjugate. Therefore every mirror has ovals of each pattern type.
- If q is not divisible 3 and $q > 7$ then all of the Macbeath-Singerman symmetries for hyperbolic $(2, 3, n)$ and $(3, 3, n)$ actions are non-separating.

Conjectures

Conjecture

All Macbeath-Singerman symmetries of $PSL_2(q)$ are non-separating.

See next slide.

Sample results

Show tables of Symmetric QP-actions

Computational Tools

We used Magma to compute all the results shown. For faster computation and for general results we can try traces in $SL_2(q)$ as follows (see Broughton et al, Glover-Sjerve and Tyszkowska).

- For each possible order n of an element in $u \in PSL_2(q)$ there is a universal polynomial $f_n(\tau) \in \mathbb{Z}[\tau]$ such that $ord(u) = n$ if and only if $f_n(Tr(U)) = 0$ for any covering element of u.
- The trace of any word in A, B, C can be easily computed as a polynomial in $\{\alpha, \beta, \gamma\}$. In particular this can be done for oval rotations.

Any Questions?

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