Approximating Coefficients of Shabat Polynomials

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Abstract

In 1984, Alexander Grothendieck, inspired by a result of Gennadii Belyĭ (1951 - 2001) from 1979, constructed a finite, connected planar bipartite graph via rational functions $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ with critical values $\{0, 1, \infty\}$ by looking at the inverse image of the triangle formed by these three points. He called such graphs Dessins d'Enfants. Conversely, Riemann's Existence Theorem implies that every finite, connected planar graph arises in this way. We are interested in constructing Shabat Polynomials (generalized Chebyshev polynomials), the Belyĭ functions corresponding to trees. This construction comes down to finding the roots of a system of nonlinear equations.

REUF4 2012

In collaboration with:

- Edray Goins (Purdue University),
- Luis Melara (Shippensburg University),
- Naomi Cameron (Lewis & Clark College),
- Emille Lawrence (University of San Francisco)
- Karoline Pershell (AAAS)

Outline of Talk

1. Belyĭ
2. Dessins
3. Shabat Polynomials
4. System of Equations

Belyĭ Maps

Let X be a Riemann surface.

In our case, $X = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \simeq S^2(\mathbb{R})$, the Riemann Sphere.

A **rational function** $\beta : X \to \mathbb{P}^1(\mathbb{C})$ is a map which is a ratio $\beta(z) = p(z)/q(z)$ in terms of relatively prime polynomials $p,\ q \in \mathbb{C}[X]$.

The **degree** of β is defined to be $\deg \beta = \max\{\deg p, \deg q\}$.

If $|\beta^{-1}(w)| < \deg(\beta)$, then w is a critical value.

Examples

A rational function $\beta : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values $\{0, 1, \infty\}$ is called a **Belyĭ map.**

$\beta : X \to \mathbb{P}^1(\mathbb{C})$

$\beta(z) = z^3$ has as critical values $\{0, \infty\}$.

$\beta(z) = \dfrac{z^3}{z^3 - 1}$ has as critical values $\{0, 1\}$.

Dessin d'Enfant

To each Belyĭ map we can associate a Dessin d'Enfant.

A "Child's Drawing" is a graph (V, E) such that

- loopless
- bipartite, $V = B \cup W$
- finite set of vertices and edges
- connected, i.e., every pair of distinct vertices is connected

Dessin d'Enfant

$$B = \beta^{-1}(\{0\}),\ W = \beta^{-1}(\{1\}),\ E = \beta^{-1}([0,1]),\ F = \beta^{-1}(\{\infty\})$$

The graph can be embedded in X without crossings.

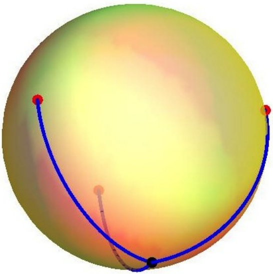
The number of vertices, edges, and faces is

$$v = |\beta^{-1}(\{0,1\})|, \quad e = \deg(\beta), \quad f = |\beta^{-1}(\{\infty\})|.$$

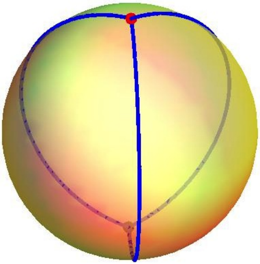
Euler's formula states that if a finite, connected, planar graph is drawn in the plane without crossing,

$$v - e + f = 2$$

Examples (images thanks to Dr. Goins)



$$\beta(z) = z^3$$



$$\beta(z) = \frac{z^3}{z^3 - 1}$$

Trees

A connected planar graph is a **tree** if it has only one face.

Euler's formula shows that a tree must have $n+1$ vertices for n edges.

The number of edges incident to a vertex are called **valencies.**

Let $\{m_0, m_1, ..., m_n\}$ denote the valencies of the black vertices.

The "degree sum formula" asserts $\sum m_i = 2n$.

We turn this tree into a bipartite graph $\Gamma = (V, E)$ by choosing the set W to be the "midpoints" of the edges.

Constructing Belyĭ maps

Find a Belyĭ map such that its Dessin d'Enfant is Γ, i.e., find two relatively prime polynomials $p(z), q(z)$ such that

- $B = \beta^{-1}(\{0\}) = \{z \in X \mid p(z) = 0\}$
- $W = \beta^{-1}(\{1\}) = \{z \in X \mid p(z) - q(z) = 0\}$
- $F = \beta^{-1}(\{\infty\}) = \{z \in X \mid q(z) = 0\}$

Each one of: $p(z)$; $p(z) - q(z)$; $q(z)$ must have a factorization over $\mathbb{C}$ and $p(z) - q(z) - (p(z) - q(z)) = 0$.

Constructing Belyĭ maps

$$p(z) = p_0(z - z_0)^{m_0}(z - z_1)^{m_1} \cdots (z - z_n)^{m_n}$$

$$p(z) - q(z) = \left(\sum_{i=0}^{n} r_i z^i \right)^2$$

$$q(z) = q_0(z - z_\infty)^{2n}$$

After a Möbius transformation, to eliminate several variables, this leads to the following theorem.

Theorem (Shabat 1994; Goins 2012)

We can find $2n-1$ complex numbers $y_1, y_2, ..., y_{2n-1}$ such that

$$z^{m_0}\,[z-1]^{m_1}\,[z-y_1]^{m_2}\cdots[z-y_{n-1}]^{m_n}+y_n^2-\left[z^n+\sum_{j=0}^{n-1} y_{j+n} z^j\right]^2=0$$

identically as a polynomial in z so that the Dessin d'Enfant of the following Belyĭ map is the aforementioned tree:

$$\beta(z)=-\frac{1}{y_n^2}z^{m_0}\,[z-1]^{m_1}\,[z-y_1]^{m_2}\cdots[z-y_{n-1}]^{m_n}\,.$$

This polynomial of degree $2n=\sum m_i$ is called a **Shabat Polynomial** or a **Generalized Chebyshev Polynomial.**

Dessins d'Enfants

Expanding:

$$z^{m_0}\left[z-1\right]^{m_1}\left[z-y_1\right]^{m_2}\cdots\left[z-y_{n-1}\right]^{m_n}+y_n^2-\left[z^n+\sum_{j=0}^{n-1}y_{j+n}z^j\right]^2$$

in y_i yields 2n-1 polynomials in $y_i, i = 1, 2, ..., 2n-1$ as coefficients for powers of z, each set equal to zero.

There are $2n-1$ equations in $2n-1$ unknowns.

Example: $n = 1$

Two black vertices, $m_0 + m_1 = 2 \Rightarrow m_0 = m_1 = 1$. Then

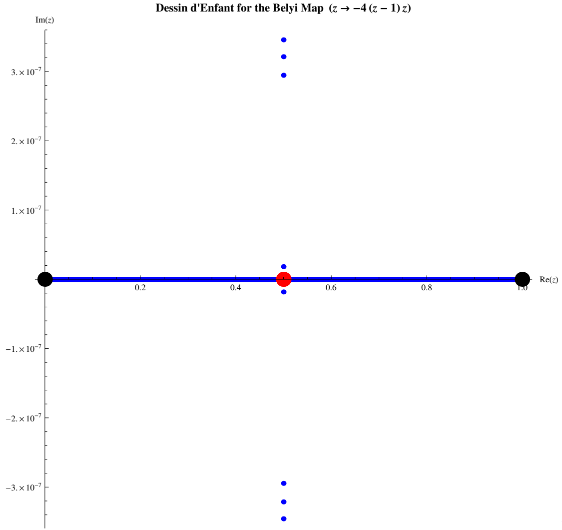
$$z(z-1) + y_1^2 - (z + y_1)^2 = 0 \Rightarrow -(1 + 2y_1)z = 0 \Rightarrow y_1 = -1/2$$

Thus $\beta(z) = -\frac{1}{y_1^2} z(z-1) = 4z(1-z)$.

Notice $\beta^{-1}(0) = \{0, 1\}, \beta^{-1}(1) = \{1/2\}, \beta^{-1}(\infty) = \{\infty\}$

and the white vertices are the midpoints of the edges.

Example: $n = 1$


Dessin d'Enfant for the Belyi Map (z → −4 (z − 1) z)
Im(z)
3. × 10^−7
2. × 10^−7
1. × 10^−7
−1. × 10^−7
−2. × 10^−7
−3. × 10^−7
0.2
0.4
0.6
0.8
1.0
Re(z)

Example: $n = 2$

Three black vertices, $m_0 + m_1 + m_2 = 4 \Rightarrow$

$\{m_0, m_1, m_2\} = \{2, 1, 1\}, \{1, 2, 1\},$ or $\{1, 1, 2\}$.

Consider $\{2, 1, 1\}$. Then

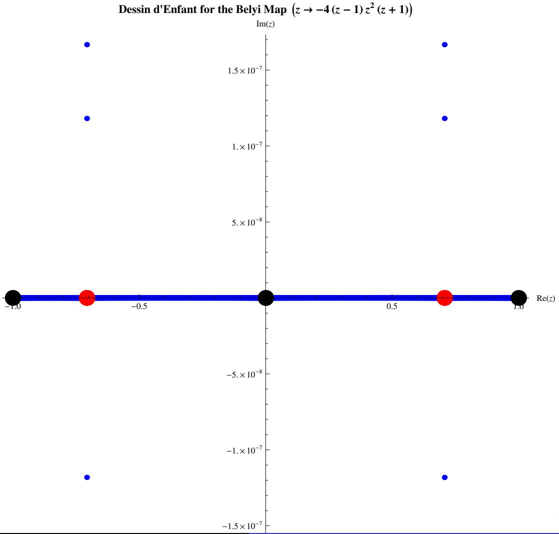
$$z^2(z-1)(z-y_1) + y_2^2 - (z^2 + y_2 + y_3 z)^2 = 0 \Rightarrow$$

$$-(y_1 + 1 + 2y_3)z^2 + (y_1 - 2y_2 - y_3^2)z - 2y_2y_3 = 0$$

Set coefficients equal to zero, $\Rightarrow \{y_1, y_2, y_3\} = \{-1, -1/2, 0\}$

$$\Rightarrow \beta(z) = 4z^2(1 - z^2)$$

Example: $n = 2$


Dessin d'Enfant for the Belyi Map $(z \to -4(z-1)z^2(z+1))$
Im(z)
Re(z)
1.5 × 10^-7
1. × 10^-7
5. × 10^-8
−5. × 10^-8
−1. × 10^-7
−1.5 × 10^-7
−1.0
−0.5
0.5
1.0

As n increases, harder to solve the system of equations.

Let's begin with a simple example.

How do we find the roots of a complicated function in one variable?

One method is Newton's Method.

More than one variable?

We generalize.

Newton's Method

$$\text{Given } x_0, \text{ do } x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots$$

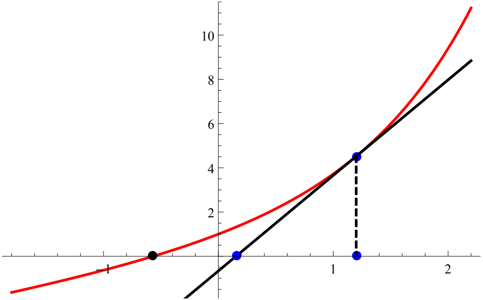


image thanks to Dr. Melara

Several Variables

Let $y = (y_0, y_1, ..., y_{2n-1})$.

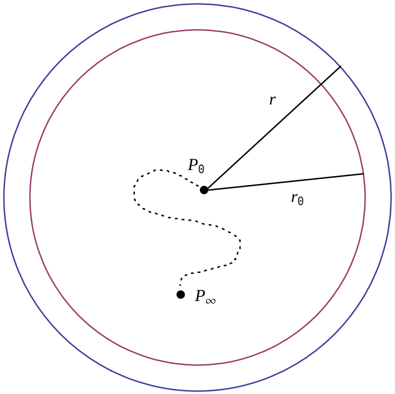
Compute the Jacobian $J(y)$ of the $2n - 1$ system of equations $F(y) = 0$.

Given initial iteration P_0,

$$P_{k+1} = P_k - J(P_k)^{-1} F(P_k), \quad k = 0, 1, 2, ...$$

If J is close to being singular, we perturb by a scalar multiple of I (Dennis and Schnabel, 1987).

Kantorovich Theorem



- J is continuously differentiable in $N(P_0, r)$
- H Lipschitz continuous in $N(P_0, r)$
- H nonsingular

Some Issues

- MATLAB vs SAGE vs Mathematica
- software could not handle complex numbers; split into real and imaginary parts
- $2(2n-1)$ equations in $2(2n-1)$ unknowns
- J is close to singular.

Sample Numerical Result; $n = 3$

Valency: $\{2, 1, 2, 1\} \Rightarrow \beta(z) = -\frac{27}{4} z^2 (z-1)(2-3z)^2(1+3z)$

Newton's Method:

$$\begin{pmatrix} 0.666666666670291 \\ 0.000000000003269 \\ -0.333333333390721 \\ 0.000000000077557 \\ 0.074074074080737 \\ -0.000000000009677 \\ -0.000000000030282 \\ 0.000000000047913 \\ -0.999999999974930 \\ -0.000000000042048 \end{pmatrix}$$

Exact:

$$\begin{pmatrix} \frac{2}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \\ \frac{2}{27} \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

Concluding Remarks

- Valency list $\{2, 1, 2, 1\}$, two Shabat Polynomials found. In general, how do you know if you have all solutions?
- Valency list $\{3, 3, 3, 1, 1, 1, 1, 1\}$, $n = 7$, found approximate solutions up to 10^{-6}.
- Not convergent for all initial iterates. How do we choose initial iterate?
- Goal: Implement code in SAGE

References

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Thank you!

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Questions?