Extremal configurations of commuting symmetries

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Preliminaries

X – compact Riemann surface;
$G = \mathrm{Aut}^{\pm}(X)$ – full automorphism group;
Symmetry σ of X – an antiholomorphic involution in G.

For σ being a symmetry of a Riemann surface X, the connected component of $\mathrm{Fix}(\sigma)$ shall be called an *oval* of σ. We call σ *separating* if $X \setminus \mathrm{Fix}(\sigma)$ is disconnected, and *nonseparating* otherwise.

For a symmetry σ on a Riemann surface of genus g we define (g, t, ε) to be the *type* of σ, where t denotes the number of ovals of σ, $\varepsilon = 0$ or 1 depending on the separability of σ.

If the genus is known, we usually just write $+t$ or $-t$.

s- and o-extremal Riemann surfaces

From the studies of Bujalance, Costa, Gromadzki, Izquierdo, Natanzon two interesting classes of surfaces emerge:

s-extremal Riemann surfaces - Riemann surfaces admitting the maximal number of nonconjugate symmetries;

o-extremal Riemann surfaces (with parameter k) - Riemann surfaces admitting the maximal total number of ovals for a system of k nonconjugate symmetries (with ovals).

Questions:

1. The group structure - done;
2. The distribution of topological types of symmetries - done in o-extremal case;
3. Real equations - done in o-extremal abelian case;
4. How can we use them?

Group structure for such extremal surfaces

Theorem (Gromadzki, EKW 2018): The automorphism group of such an extremal Riemann surface is isomorphic to the direct product of a dihedral group and some number of copies of cyclic groups of order 2.

Remark: For $k \neq 4, 5$, the automorphism group of an o-extremal surface is abelian.

Knowing the structure of the automorphism group for o-extremal surfaces, we can try to find all the topological types of symmetries and real equations for the underlying surface and its real forms (in abelian case). We can also revisit problems concerning the topological properties of the real nerve of the moduli space of Riemann surfaces

For our purposes the key results are:

Theorem (Gromadzki 2000): The maximal total number of ovals of k nonconjugate symmetries on a Riemann surface of genus g does not exceed

$$2g - 2 + (9 - k)\frac{|G|}{8},$$

where G denotes the 2-group of automorphisms generated by the symmetries.

Let now $g = 2^{r-1}u + 1$ for some odd u.

Theorem (Bujalance, Gromadzki, Izquierdo 2001): Let $\omega(g)$ denote the maximal possible number of nonconjugate symmetries on a Riemann surface of genus g. Then:
(1) $\omega(g) \leq 2^{r+1}$;
(2) $\omega(g) = 2^{r+1} \Leftrightarrow u \geq 2^{r+1} - 3$;
(3) the remaining values of $\omega(g)$ were calculated.

Numbers of ovals

Theorem TS: If we have an o-extremal configuration of 3 symmetries on a Riemann surface of genus g, then they commute and their types, for integers $\alpha, \beta > 0$ such that $2\alpha + 2\beta = g + 3$ are

$$+(g+1), +2\alpha, +2\beta.$$

All possible configurations are realized.

Theorem T: Let $\sigma_1, \ldots, \sigma_k$ for $k \geq 4$ be commuting symmetries, which generate the group $\mathbb{Z}_2^A$, on an o-extremal Riemann surface of genus g. Then $\frac{g-1}{2^{A-2}} \geq k-3$ is an integer , one of the symmetries has $g - 1 + (5-k)2^{A-3}$ ovals while the other have $\alpha_i 2^{A-3}$ ovals each, where $\sum \alpha_i = \frac{g-1}{2^{A-2}} + 2$. All the possible configurations are realized.

Separability types

Prop S1: For any $k \geq 4$, the symmetries generating the group $\mathbb{Z}_2^k$ realize the maximal possible total number of ovals on a Riemann surface of genus g if and only if all of them are separating and hold the conditions of the Theorem T.

Prop S2: For $k \geq 9$, the symmetries generating the group $G = \mathbb{Z}_2^r$, where r is the smallest possible positive integer such that $k \leq 2^{r-1}$, realize the maximal possible total number of ovals on a Riemann surface of genus g if and only if all the symmetries are nonseparating and hold the conditions of Theorem T.

Prop S3: 9 symmetries in the group $G = \mathbb{Z}_2^A$, where $A = 6, 7, 8$, realize the maximal possible total number of ovals on a Riemann surface of genus g if and only if:

1. at least 7 of them are nonseparating for $A = 6$;
2. at least 6 of them are nonseparating for $A = 7$;
3. $4, 6$ or 8 are nonseparating for $A = 8$. All the possible values are realized, as far as the conditions of the Theorem T hold.

Nerve $\mathcal{N}(g)$ of $\mathcal{M}_g^{\mathbb{R}}$

$\mathcal{M}_g^{\mathbb{R}} = \bigcup_{(g,t,\varepsilon)} \mathcal{M}_g^{t,\varepsilon}$

$\mathcal{M}_g^{\mathbb{R}} \longmapsto \mathcal{N}(g)$ - the nerve of $\mathcal{M}_g^{\mathbb{R}}$ (Čech simplicial complex)

points in $\mathcal{N}(g) \longleftrightarrow$ strata $\mathcal{M}_g^{t,\varepsilon}$ in $\mathcal{M}_g^{\mathbb{R}}$

$$(\mathcal{M}_g^{t_1,\varepsilon_1}, \ldots, \mathcal{M}_g^{t_n,\varepsilon_n}) \text{ a simplex } \Leftrightarrow \mathcal{M}_g^{t_1,\varepsilon_1} \cap \ldots \cap \mathcal{M}_g^{t_n,\varepsilon_n} \neq \emptyset$$
$$\Leftrightarrow \begin{cases} \text{there exists a Riemann surface } X \text{ having } n \text{ symmetries of types } (g, t_1, \varepsilon_1), \ldots, (g, t_n, \varepsilon_n). \end{cases}$$

Problem: Find the properties of $\mathcal{N}(g)$.

What is known?

Obviously, the geometrical dimension of $\mathcal{N}(g)$ is limited by the maximal number of nonconjugate symmetries.

Theorem (Gromadzki, EKW 2011-2016): Let $g = 2^{r-1}u + 1$ where u is odd. Then:

(1) If $r = 1$, which means that g is even, then $\dim_{\mathrm{G}}(\mathcal{N}(g)) = 3$;

(2) If $u \geq 2^{r-1}(2^{r+1} + 2) - 5$ then $\dim_{\mathrm{G}}(\mathcal{N}(g)) = 2^{r+1} - 1$.

We also conjectured that for odd g this is also the necessary condition on u. But it turned out to be false.

Theorem (Sikorski, EKW): Let $g = 2^{r-1}u + 1$, odd u, Then a Riemann surface of genus g has 2^{r+1} nonconjugate symmetries such that each of the symmetries has a distinct topological type if and only if $u \geq u_{min} = 2^{r-1}(2^{r+1} + 1) - 3$.

Corollary: For odd values of g, $\dim_G(\mathcal{N}(g)) = 2^{r+1} - 1$ if and only if $u \geq u_{min}$.

Some new results:

Theorem: Let $g = 2^{r-1}u + 1$ for some odd u and $r \geq 2$. Let also $a \leq r - 1$ be a natural number. Then for $u \geq 2^{2r} + 2^r - 3$ the homology groups $H_{2^{a+1}-1}(\mathcal{N}(g))$ are nontrivial.

The proof of this theorem is, again, based on the surgery of the sole nonempty period cycle, aimed at obtaining an empty simplex in the particular homology group.

This requires constructing the cycle composed of 2^{a+1} Riemann surfaces. Here, we use the o-extremality in a clever way.

Namely, one of the faces of the empty simplex that we build is an o-extremal Riemann surface, admitting already a fixed point free symmetry in its configuration.

This means, that it is not possible to add any more symmetries to the configuration, as that would exceed the maximal possible total number of ovals.

Therefore we are guaranteed, that the face corresponding the o-extremal Riemann surface is not a face of a higher dimensional simplex.

Hence we indeed get an empty simplex in the nerve and, as a result, we obtain a generator in the associated homology group. The fact, that this idea only works for the homologies of degree $2^{a+1} - 1$ comes from the fact, that the function $\omega(g, k)$, mentioned before, becomes decreasing for $k \geq 8$.

To make sure that we cannot add any more symmetries in the automorphism group, we need to have, counting the fixed point free symmetry, exactly 2^{a+1} symmetries. This guarantees, that with the growth of k and fixed g, the value of $\omega(g, k)$ decreases, and so the o-extremal simplex we constructed is not a face of a higher dimensional simplex.

Theorem: For $g \geq 4$ being even, in the $\mathcal{N}(g)$ there always exists a lonely edge defined by the types $\{-2, +(g-1)\}$. It is not a face of any triangle, but its vertices are parts of higher dimensional simplices.

Thus we have

Corollary: For any even $g \geq 4$ we have $H_1(\mathcal{N}(g)) \neq 1$.

This result seems to be connected to a phenomenon that the types are more likely to be joined by an edge in $\mathcal{N}(g)$ is they share parity or sign. The lonely edge is an example that breaks that rule and seems to be the only one for small values of g.

We can prove even more,

Theorem: For any even $g > 4$ all the homology groups of $\mathcal{N}(g)$ are nontrivial.

The idea of the proof that $H_2(\mathcal{N}(g)) \neq 1$ once again asks for constructing the necessary cycle of Riemann surfaces. This is done by examining the sequence of types

$$-g, -(g-2), -3, -2.$$

We construct four surfaces that admit commuting symmetries of the types:
$\{-g, -(g-2), -3\}$,
$\{-g, -(g-2), -2\}$,
$\{-g, -3, -2\}$,
$\{-(g-2), -3, -2\}$.

Then we show that the full tetrahedron with all these types cannot exist, as together the symmetries have $2g+3$ ovals, which exceeds the maximal possible value for four symmetries on a Riemann surface of even genus, which is $2g+2$. Moreover, this cycle cannot be a boundary which completes the proof.

The real equations

Theorem: For any choice of the real numbers $b_1 < b_2 < \ldots < b_g < 0$, the equations

$$y^2 = (x - b_{2\alpha+1}) \ldots (x - b_g)x(x-1),$$
$$z^2 = (x - b_1) \ldots (x - b_{2\alpha})$$

define a complex algebraic curve of an odd genus g, admitting three real forms corresponding to commuting symmetries on a respective o-extremal Riemann surface. Conversely, if a Riemann surface of odd genus g admits an o-extremal configuration of 3 symmetries, then its equations are of the form above for integers g and some $0 < \alpha \leq (g+1)/2$.

Theorem: For any choice of the real numbers $b_1 < b_2 < \ldots < b_g < 0$, the equations

$$y^2 = (x - b_{2\alpha+2}) \ldots (x - b_{2\alpha+2\beta+2\gamma})x(x-1),$$
$$z^2 = (x - b_1) \ldots (x - b_{2\alpha+2\beta+2})$$

define a complex algebraic curve of an even genus g, where $2\alpha + 2\beta + 2\gamma = g$, admitting three real forms corresponding to the three commuting symmetries on a respective o-extremal Riemann surface. Conversely, if a Riemann surface of an even genus g admits such an o-extremal configuration of symmetries, then its defining equations are of the form above for some integers $\alpha, \beta, \gamma \geq 0$ such that $2\alpha + 2\beta + 2\gamma = g$.

This results were generalized for the case of at least 4 commuting symmetries in the recent paper coauthored with Peter Turbek:

Real equations for o-extremal Riemann surfaces with abelian automorphism groups, Glasnik Matematički 60 (2) (2025), 267-290.

The result is too technical to present it here, but describes precisely the real defining equations for any *o*-extremal Riemann surface, under the assumption that the group generated by the symmetries is abelian. In the last part od the paper, we identify which of the symmetries corresponds to the complex conjugation, we provide its real equations and we repeat the procedure for all the other symmetries to achieve the most complete result.

Thank you for your attention!