Some Equations of Curves $g = 10$ with big automorphism group

Motivation

- In 2002 MSSV produced a table of all the possible Automorphism groups up to $g = 10$.

- Genus 10, $\delta = 0$

1	(432,734)	$(2,3,8)$	**2**	(432,734)	$(2,3,8)$
3	(360,118)	$(2,4,5)$	**4**	(324,160)	$(2,3,9)$
5	(216,92)	$(2,3,12)$	**6**	(216,158)	$(2,4,6)$
7	(216,87)	$(2,4,6)$	**8**	(216,153)	$(3,3,4)$
9	(180,19)	$(2,3,15)$	**10**	(168,42)	$(2,4,7)$
11	(162,14)	$(2,3,18)$	**12**	(144,122)	$(2,3,24)$
13	(108,25)	$(2,6,6)$	**14**	(108,15)	$(2,4,12)$
15	(88,7)	$(2,4,22)$	**16**	(80,6)	$(2,4,40)$
17	(72,28)	$(2,6,12)$	**18**	(72,23)	$(2,6,12)$
19	(72,42)	$(3,4,6)$	**20**	(63,3)	$(3,3,21)$
21	(60,10)	$(2,6,30)$	**22**	(42,6)	$(2,21,42)$
23	(42,2)	$(3,6,14)$	**24**	(42,2)	$(3,6,14)$

Figure: Automorphism table

- I produce equations for curves with id's **3,4**.

The curve with $\mathbf{Id} = 3$

- The curve with $\mathbf{id} = 3$ was well known to the classics
- The group of 360 automorphisms is easily seen to be the group A_6
- Wiman wrote the equation for this curve

[1] $$10x^3y^3 + 9z(x^5 + y^5) - 45z^2x^2y^2 - 135z^4xy + 27z^6 = 0 \quad (1)$$

- Wiman showed that there are no double points and the form is invariant under the action of A_6.
- Use the genus $\frac{(6-1)(6-2)}{2} = 10$

[1] Crelle's Vol.47, (1896) p.553

The curve with **id** $= 4$

Theorem

Let $\Gamma(9)$ *be the congruence subgroup of level* 9 *in* $PSL_2(\mathbb{Z})$, *That is the matrices congruent to the identity matrix modulo* 9. $PSL_2(\mathbb{Z}_9)$ *has an order of* 324 *and* $\mathbb{H}/\Gamma(9)$ *is of genus* 10 *with an automorphism group of* 324 *elements.*

- Because of the uniqueness of this curve according to MSSV $\mathbb{H}/\Gamma(9)$ must be the curve of **id=4**
- We can easily identify the elements that create the $(2,3,9)$ triple coming from $PSL_2(\mathbb{Z})$
- The element of order 9 is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Equations for $\mathbb{H}/\Gamma(9)$

Theorem

$X(9) = \overline{\mathbb{H}/\Gamma(9)}$ is given as a complete intersection of 2 surfaces in $\mathbb{CP}^3$

$$\begin{aligned} -x_1x_3^2 + x_1^2x_4 + x_3x_4^2 &= 0 \\ -x - 2^3 + x_1^2x_3 - x_1x_4^2 + x_3^2x_4 &= 0 \end{aligned} \tag{2}$$

where $(x_1 : x_2 : x_3 : x_4)$ is a point in $\mathbb{CP}^3$.

$PSL_2(\mathbb{Z}_9)$ is the automorphism group of this intersection.

Sketch of proof

- Use one dimensional theta functions:
  $\theta \begin{bmatrix} \frac{1}{9} \\ 1 \end{bmatrix} (0,\tau), \theta \begin{bmatrix} \frac{3}{9} \\ 1 \end{bmatrix} (0,\tau), \theta \begin{bmatrix} \frac{5}{9} \\ 1 \end{bmatrix} (z,\tau), \theta \begin{bmatrix} \frac{7}{9} \\ 1 \end{bmatrix} (z,\tau)$, $\tau \in \mathbb{H}$
  the upper half plane to define a mapping: $\mathbb{H} \mapsto \mathbb{C}^4$
- Using a transformation formula for theta functions it turns out that this mapping induces a mapping : $\phi : \mathbb{H}/\Gamma(9) \mapsto \mathbb{CP}^3$ under the action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = \frac{a\tau + b}{c\tau + d}$$

- Define

$$x_i = \theta \begin{bmatrix} \frac{2i-1}{9} \\ 1 \end{bmatrix} (0,\tau)$$

Sketch of Proof

- The mapping $\phi : \mathbb{H}/\Gamma(9) \mapsto \mathbb{CP}^3$ extends to compactification of $\phi : \overline{\mathbb{H}/\Gamma(9)} \mapsto \mathbb{CP}^3$
- To find relations between theta functions use the residue theorem for elliptic curves: For example :

Example

$$\frac{\theta^2 \begin{bmatrix} \frac{1}{9} \\ 1 \end{bmatrix} (z,\tau)\theta \begin{bmatrix} \frac{7}{9} \\ 1 \end{bmatrix} z,\tau)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0,\tau)\theta \begin{bmatrix} \frac{3}{9} \\ 1 \end{bmatrix} (0,\tau)\theta \begin{bmatrix} \frac{15}{9} \\ 1 \end{bmatrix} (z,\tau)} \tag{3}$$

produces the identity: $-x_1x_3^2 + x_1^2x_4 + x_3x_4^2 = 0$ by computing the residues of the elliptic functions and summing to 0.

Finishing the proof

- The complete intersection of the surfaces is the curve of genus 10 using the formula $\frac{1}{2}ab(a+b-4)+1$.
- We have a mapping from one genus 10 curve to another genus 10 curve thus it must be onto and $1-1$.

Differentials

Using the embedding we can explore the analytical properties of the curve.

Theorem

The canonical embedding of $\overline{\mathbb{H}/\Gamma(9)}$ *is given by:* $x_i x_j$ *into* $\mathbb{CP}^9$. *In terms of modular forms we have:*

$$\eta^2(\tau)\theta\begin{bmatrix}\frac{2i-1}{9}\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}\frac{2j-1}{9}\\1\end{bmatrix}(0,\tau)$$

Summary

- We have identified two curves in the MSSV table
- In the case of $X(9) = \mathbb{H}/\Gamma(9)$ the equation os obtained explicitly through theta series
- The groups of these curves tend to be linear so maybe one can use invariant theory to get explicit equations.