Quasisimple groups that act with almost all signatures

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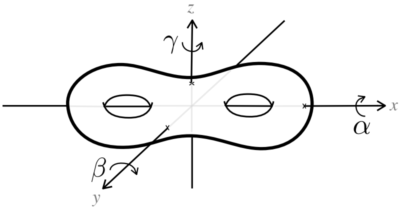
Overarching Research Question

Question

For a given finite group G, can we describe all the distinct G-actions on compact Riemann surfaces with genus at least 2? What about for a specific genus $\sigma \geq 2$?



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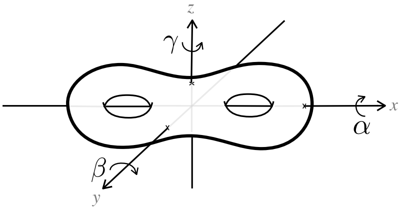

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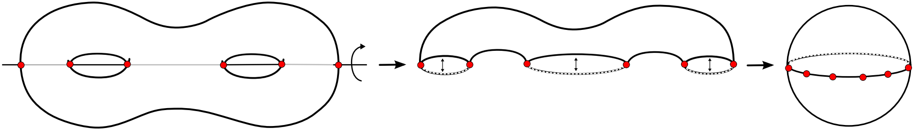

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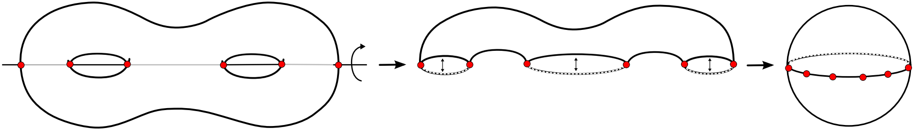

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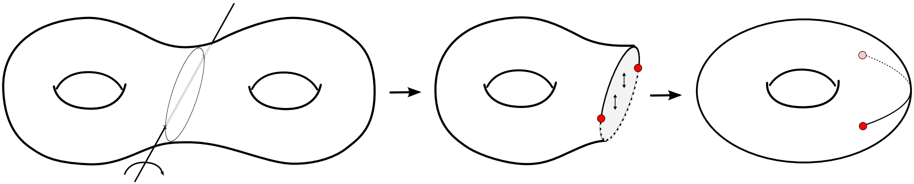

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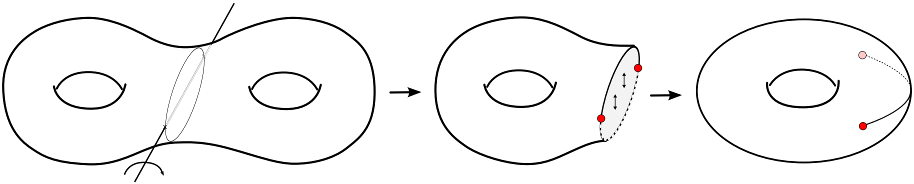

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Refining the Initial Question

Question

For a given finite group G, can we determine all signatures with which G acts on compact Riemann surfaces of genus at least 2? Can we determine the signatures for a fixed genus $\sigma \geq 2$?

Necessary Numerical Conditions for Signatures

Fact

If G acts on a compact Riemann surface S with signature $(h; m_1, \ldots, m_r)$, then each m_i is the order of a non-trivial element in G.

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If G acts with signature $(h; m_1, \ldots, m_r)$ on a surface S with genus $\sigma \geq 2$, then the Riemann-Hurwitz formula holds:

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Potential signatures for C_6 are of the form

$$(h; 2, \ldots, 2, 3, \ldots, 3, 6, \ldots, 6)$$

For potential signatures of C_6 in genus $\sigma = 6$, we solve:

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(Caspary, W) All quasisimple groups are AAS with the exception of the degree 2 cover of the projective special linear group $L_2(q)$ for each odd q, the double covers of the alternating groups A_5, A_6 and A_7 and the sixfold covers of A_6 and A_7.

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     - This group cannot be contained in the center, so must be all of G.

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