# Cliques in regular maps

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All regular maps on non-orientable surfaces are reflexible. On orientable surfaces, the orientation-preserving automorphisms form a subgroup $Aut^+(M)$ of index at most two in $Aut(M)$. If an orientable regular $M$ is not reflexible, $Aut(M) = Aut^+(M)$ and $M$ is called **chiral**.

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# Angle measure

Given vertex $v$ and adjacent vertices in local cyclic order $u_1, \ldots u_n$, the **measure** of angle $u_i v u_j$ is the smaller of $|i-j|$ or $n-|i-j|$. In particular the measure is at most $n/2$.

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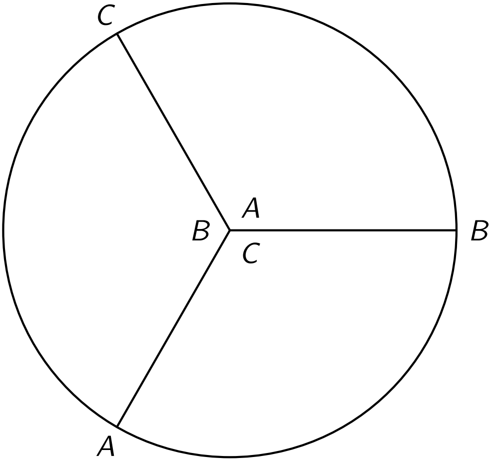
Any automorphism of a regular map preserves angle measure Moreover, if the map is reflexible, every triangle is equiangular by local dihedral symmetry at every vertex.

Let $v, u_1, u_2, u_3$ be vertices of a clique of size 4. Let the three angles formed at $v$ have measures $A \leq B \leq C$. Let $n$ be the valence of $v$. There are two possibilities:
1) $A+B+C=n$,
2) $A+B=C$.


C
A
B
C
B
A

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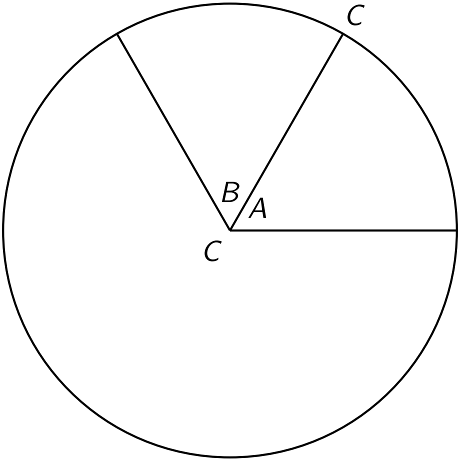
Suppose the clique was contained in a clique of size 5. The fourth edge at $v$ could not make angle $n/3$ with other three edges at $v$, so there would be a 4-clique containing $v$ whose angles were not all $n/3$. We conclude that if there is a 4-clique with $A + B + C = n$, there can be no larger cliques.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Lower triangle has all angles $C$ as does the outer triangle. We conclude that the angles at right have measure $A, C, C$ so $A + 2C = n$. In the same way, the angles at upper left have measures $C, C, B$ so $B + 2C = n$.

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Again by the angle restrictions, there cannot be a clique of size 7 containing $K$.

# Non-orientability for $K_6$

Suppose $M$ is orientable. Let $G$ be the subgroup of $Aut(M)$ taking $K_6$ to $K_6$. As a permutation group, it is a subgroup of the symmetric group $S_6$. Since vertex stabilizers have order 10 (dihedral group $D_5$), we have $|G| = 60$.

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$G$ is orientation-reversing. Let $H$ be its index two orientation-preserving subgroup. It contains 5-fold rotations around each vertex giving 24 even permutations of order 5, which therefore generate $H$. We conclude $H$ is a subgroup of the alternating group $A_6$.

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Since $|H| = 30$, it contains an involution, but every even involution in $S_6$ must leave two vertices fixed. The only map automorphism leaving two vertices fixed is reflection in an edge, a contradiction since $H$ is orientation-preserving. We conclude that $M$ is non-orientable.

# Canonical generators and balanced Cayley maps

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**Algebraic definition of a regular map**: Group $G$ with *canonical* generators $y, z$ where $z^2 = 1$. Vertices cosets of $\langle y \rangle$, edges cosets of $\langle z \rangle$ and faces cosets of $\langle yz \rangle$ and incidence is non-empty intersection. Not just the group $G$ but also choice of $y, z$.

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# Chiral maps for $K_n$

**Theorem** (Biggs 1971). Let $c$ generate the cyclic multiplicative group of the finite field $\mathbb{F}(q)$. Let $\phi : \mathbb{F} \to \mathbb{F}$ be the automorphism $\phi(u) = cu$. Then the balanced Cayley map for $\phi$ is a chiral regular map with underlying graph $K_q$.

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**History** Heffter 1898 for $K_p$.
Chiral maps: none for genus $0, 2, 3, 4, 5, 6$. Coxeter and Moser (1955) once conjectured the only chiral maps are in the torus. Edmonds 1960 map for $K_8$, genus 7. (Also Frucht 1952?)

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We conclude that $y^i$ and $z$ are canonical generators for a regular map with underlying graph $K$, forcing $K$ to have size $p^e$.

# Edge-disjoint example with smallest valence

Let $V$ be the vector space of dimension 2 over the finite field $\mathbb{F}(q)$. Let $\phi : V \to V$ be the linear transformation with matrix $[0\,1 \,|\, c\,0]$ where $c$ generates $\mathbb{F}^*(q)$. Let $u = [1\,0]$. The orbit of $u$ generates $V$ and includes $c^{(q-1)/2}u = -u$ when $q$ is odd. Let $M$ be the associated balanced Cayley map.

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4) These two cliques are edge-disjoint as 1-dimensional subspaces of $V$.

# Paley maps

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**Exercise** Show that $c = 2$ generates $\mathbb{F}^*(29)$ and that the powers of $c^2 = 4$ are $\pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 13$.

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# Paley maps

Given $q \equiv 1 \bmod 4$ and $c$ generating $\mathbb{F}^*(q)$, let $\phi : \mathbb{F}(q) \to \mathbb{F}(q)$ be $\phi(u) = c^2 u$. Since $q \equiv 1 \bmod 4$ , we have $(c^2)^{(q-1)/4} = c^{(q-1)/2} = -1$ so the orbit of any $u$ is inverse-closed. Let $P(q)$ be the corresponding balanced Cayley map, called the **Paley map** for $\mathbb{F}(q)$ (corresponding graph is the **Paley graph**).

**Exercise** Show that $c = 2$ generates $\mathbb{F}^*(29)$ and that the powers of $c^2 = 4$ are $\pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 13$. Show $0, 4, 5, 9$ are vertices of a clique of size 4. Show maximal cliques in $P(29)$ cannot be edge-disjoint (Hint: Count edges in $P(29)$.or show $-4, 0, 1, 5$ is also a clique and shares edge 05)

I have emailed everyone a list of $\omega(P(p))$ for all $p \leq 16729$. No gaps up to $\omega(P) = 23$ (lots that are not prime powers). In table $log_2(p) \leq \omega \leq log_2(p)$. Known $\omega \leq \sqrt{p}$, little else. Connections to Ramsey theory, designs ... Everyone has worked on this: Erdös, Tao, Brouwer, Gleason, Praeger, Jones ...