Curves with automorphisms in Mukai's models of the moduli spaces of curves

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Hurwitz curves

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Can we find examples?

- Genus 2: there are no Hurwitz curves; the largest automorphism group is order 48 for $y^2 = x^6 + 1$.
- Klein 1879: $x^3y + y^3z + z^3x = 0$ is a genus 3 Hurwitz curve.
- Wiman 1895: in genus 4, 5, 6, there are no Hurwitz curves.
- Fricke 1899: there is a genus 7 Riemann surface with $|\operatorname{Aut}(C)| = 84(g-1) = 504$.
  - Macbeath 1965: explicit equations of the genus 7 curve with 504 automorphisms.
- The next Hurwitz curves are in genus 14.
- There are infinitely many Hurwitz curves, but they are increasingly rare as g grows.

The Fricke-Macbeath curve in genus 7

Macbeath's equations:

$$
\begin{gathered}
y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2, \\
\sum_{i=0}^{7} \zeta_7^i y_i^2 \\
\sum_{i=0}^{7} \zeta_7^{-i} y_i^2 \\
-a_3 y_0 y_6 + a_2 y_1 y_4 + a_1 y_3 y_5, \\
-a_3 y_1 y_0 + a_2 y_2 y_5 + a_1 y_4 y_6, \\
-a_3 y_2 y_1 + a_2 y_3 y_6 + a_1 y_5 y_0, \\
-a_3 y_3 y_2 + a_2 y_4 y_0 + a_1 y_6 y_1, \\
-a_3 y_4 y_3 + a_2 y_5 y_1 + a_1 y_0 y_2, \\
-a_3 y_5 y_4 + a_2 y_6 y_2 + a_1 y_1 y_3, \\
-a_3 y_6 y_5 + a_2 y_0 y_3 + a_1 y_2 y_4
\end{gathered}
$$

where $a_j = \zeta_7^j - \zeta_7^{-j} = 2i \sin\left(\frac{2\pi j}{7}\right)$.

Mukai's approach to genus 7 canonical curves

Mukai: for a general smooth genus 7 curve (no g^1_2, g^1_3, or g^1_4),

$$\mathrm{Sym}^2(I_2) \to I_4$$

has one-dimensional kernel. Let Q be a generator.
$\Rightarrow (I_2^\vee, Q)$ is a quadratic vector space.

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For each $p \in C$, the row space of the Jacobian matrix

$$\left[\frac{\partial f_j}{\partial y_i}(p)\right]_{i=0,\dots,6}^{j=1,\dots,10}$$

is a Lagrangian of $(I_2^\vee, Q)$, denoted $W_p^\perp$

The orthogonal Grassmannian $\mathrm{OG}(5, 10)$

For any n, the orthogonal Grassmannian $\mathrm{OG}(n, 2n) \subset \mathrm{Gr}(n, 2n)$ parametrizes Lagrangian subspaces of a quadratic vector space.

We have $\mathrm{OG}(5, 10) \hookrightarrow \mathbb{P}^{15}$. Mukai's coordinates are x_0, x_{ij}, and x_{ijkl} for subsets of $\{1, 2, 3, 4, 5\}$, and the equations of $\mathrm{OG}(5, 10)$ are

$$
\begin{gathered}
x_0x_{2345} - x_{23}x_{45} + x_{24}x_{35} - x_{25}x_{34}, \\
x_{12}x_{1345} - x_{13}x_{1245} + x_{14}x_{1235} - x_{15}x_{1234}, \\
x_0x_{1345} - x_{13}x_{45} + x_{14}x_{35} - x_{15}x_{34}, \\
x_{12}x_{2345} - x_{23}x_{1245} + x_{24}x_{1235} - x_{25}x_{1234}, \\
x_0x_{1245} - x_{12}x_{45} + x_{14}x_{25} - x_{15}x_{24}, \\
x_{13}x_{2345} - x_{23}x_{1345} + x_{34}x_{1235} - x_{35}x_{1234}, \\
x_0x_{1235} - x_{12}x_{35} + x_{13}x_{25} - x_{15}x_{23}, \\
x_{14}x_{2345} - x_{24}x_{1345} + x_{34}x_{1245} - x_{45}x_{1234}, \\
x_0x_{1234} - x_{12}x_{34} + x_{13}x_{24} - x_{14}x_{23}, \\
x_{15}x_{2345} - x_{25}x_{1345} + x_{35}x_{1245} - x_{45}x_{1235}.
\end{gathered}
$$

Mukai's approach to genus 7 canonical curves

Theorem [Mukai, 1995]

- When C is a general smooth genus 7 curve,

$$\begin{array}{rccccc} \text{Spin}: & C & \to & \mathrm{OG}(5,10) & \to & \mathbb{P}^{15} \\ & p & \mapsto & W_p^{\perp} & & \end{array}$$

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  is an embedding
- $\text{Spin}(C) = P \cap OG(5,10)$, where $P \cong \mathbb{P}^6 \subset \mathbb{P}^{15}$

Challenge: Write a linear space P such that $P \cap OG(5,10)$ is the Fricke-Macbeath curve.

Algorithm 1: Compute spinors associated to points

Inputs

1. Quadrics defining the canonical ideal of C
2. 7 points p_i on C in general position

Output

1. A linear space P such that $P \cap \mathrm{OG}(5, 10) \cong C$

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3. $P = \mathrm{Span}(s_1, \ldots, s_7)$.

Enhanced algorithm 1: Compute spinors associated to points

Inputs

1. Quadrics defining the canonical ideal of C
2. 8 points p_i on C in general position

Outputs

1. A linear space P and an isomorphism $F : P \cap \mathrm{OG}(5, 10) \to C$

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4. Compute $F \in \mathrm{PGL}(7)$ mapping $s_i \mapsto p_i$.

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Results

- I implemented Algorithm 1 in `Macaulay2`
- I computed over $\mathbb{Q}[\zeta_{28}]$
  - Macbeath's equations have ζ_7 in the coefficients
  - I also wanted i to change the basis to go from the quadratic form
$$q_0^2 + q_1^2 + \cdots + q_9^2$$
    to
$$q_0 q_5 + q_1 q_6 + q_2 q_7 + q_3 q_8 + q_4 q_9$$
- For the input points, I used some of the fixed points of automorphisms. These are defined over $\mathbb{Q}[\zeta_7]$
- The calculation is correct, but the output is long

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Strategy 2: Use the automorphisms

1. Quadrics defining the canonical ideal of C
2. Generators $g_i \in \mathrm{GL}(7)$ of $\mathrm{Aut}(C)$

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Steps

1. Compute $\ker(\mathrm{Sym}^2 I_2 \to I_4)$ and "diagonalize" the resulting quadratic form.
2. Compute the action of each g_i on these new quadrics.
3. Lift these actions to Spin(10) and compute their actions on the half-spin representation S^+. Get a double cover of $\mathrm{Aut}(C)$.
4. This 16-dimensional representation decomposes as the sum of two irreducible representations of dimensions 7 and 9. Project to the 7-dimensional submodule.

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Results

- I implemented Strategy 2 in `Macaulay2` and `Magma`
- The output agrees with that of Algorithm 1

Working over other fields

We briefly discuss further calculations over $\mathbb{F}_p$ and $\mathbb{Q}$.

Working over $\mathbb{F}_p$

Mukai's results hold over any algebraically closed field, and Algorithm 1 makes sense over any field.

I ran Algorithm 1 over $\mathbb{F}_{113}$. Everything works!

Hyperplanes yielding the Fricke-Macbeath curve:

$$\begin{aligned}
&15x_{13} + 15x_{14} - x_{23} - x_{24},\\
&-39x_0 - 46x_{12} + 51x_{13} - 56x_{14} - 48x_{15} + 11x_{23} + 39x_{25} - x_{34},\\
&-47x_0 + 2x_{12} + 26x_{13} + 50x_{14} + 7x_{15} + 36x_{23} + 17x_{25} - x_{35},\\
&35x_0 + 15x_{12} + x_{13} + x_{14} + x_{15} + 15x_{23} + 40x_{25} - x_{45},\\
&46x_0 - 39x_{12} + 53x_{13} - 49x_{14} + 39x_{15} - 6x_{23} + 48x_{25} - x_{1234},\\
&2x_0 + 47x_{12} - 36x_{13} + 41x_{14} - 17x_{15} + 24x_{23} + 7x_{25} - x_{1235},\\
&15x_0 - 35x_{12} - 15x_{14} - 40x_{15} + x_{25} - x_{1245},\\
&41x_0 + 12x_{12} - 30x_{13} + 8x_{14} - 39x_{15} + 39x_{23} + 46x_{25} - x_{1345},\\
&12x_0 - 41x_{12} + 32x_{13} - 7x_{14} - 46x_{15} + 38x_{23} - 39x_{25} - x_{2345}
\end{aligned}$$

Working over $\mathbb{Q}$

There are two known models of the Fricke-Macbeath curve with coefficients in $\mathbb{Q}$ due to Brock 2004 and Hendriks 2013. (Thanks, Top and Verschoor!)

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Challenge: Write a linear space P such that $P \cap \mathrm{OG}(5, 10)$ yields these models.

Brock's model

Brock's model of the Fricke-Macbeath curve:

$$
\begin{aligned}
&y_0y_2 + 12y_3^2 - y_4y_6,\\
&-y_1^2 + y_0y_3 - 2y_5y_6,\\
&y_0y_4 + 16y_3y_5 + 8y_6^2,\\
&-y_1y_3 + y_0y_5 + \tfrac{1}{2}y_2y_6,\\
&-y_2y_3 + 2y_5^2 + y_0y_6,\\
&y_1y_2 + 12y_3y_5 + 4y_6^2,\\
&-2y_2y_3 + y_1y_4 - 8y_5^2,\\
&-y_3^2 + y_1y_5 + \tfrac{1}{4}y_4y_6,\\
&-\tfrac{1}{2}y_3y_4 - \tfrac{1}{2}y_2y_5 + y_1y_6,\\
&y_2^2 + 2y_4y_5 + 8y_3y_6.
\end{aligned}
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2. Compute the spinors s_i of the Lagrangians associated to the points p_i. Brock's model only has 2 points over $\mathbb{Q}$. Need to work over $\mathbb{Q}[\zeta_7]$ to get more.

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4. Compute $F \in \mathrm{PGL}(7)$ mapping $s_i \mapsto p_i$.
   F is defined over $\mathbb{Q}$ as well

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Brock's model in Mukai's setup

Theorem (S., 2026) The following equations yield Brock's model of the Fricke-Macbeath curve in Mukai's model of $\overline{M}_7$.

$$
\begin{aligned}
x_0 &= y_0 & x_{12} &= -32x_{2345}\\
x_{45} &= \tfrac{1}{16}y_1 & x_{13} &= -16x_{45}\\
x_{1234} &= \tfrac{1}{2}y_6 & x_{14} &= x_{1235}\\
x_{1235} &= \tfrac{1}{2}y_5 & x_{15} &= 0\\
x_{1245} &= -\tfrac{1}{32}y_4 & x_{23} &= 0\\
x_{1345} &= -\tfrac{1}{16}y_3 & x_{24} &= 4x_{1245}\\
x_{2345} &= \tfrac{1}{32}y_2 & x_{25} &= -8x_{1345}\\
 & & x_{34} &= 12x_{1345}\\
 & & x_{35} &= \tfrac{1}{2}x_{1234}
\end{aligned}
$$