Genera of smooth quotients of triangle groups by means of uniform hypermaps

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If one of the three parameters, say, r, is equal to 2, then $\mathcal{H}$ is a *map*, of type (p, q), with monodromy group G; such a map is *uniform* if all its face boundary walks have length p and all vertices have valency q.

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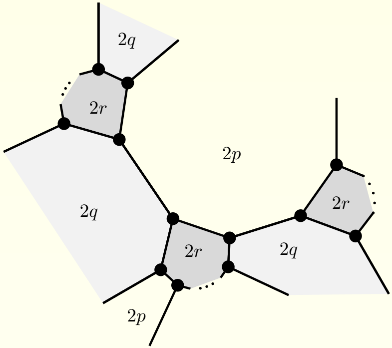
James representation $J(\mathcal{H})$ of a uniform hypermap $\mathcal{H}$ of type (p, q, r):



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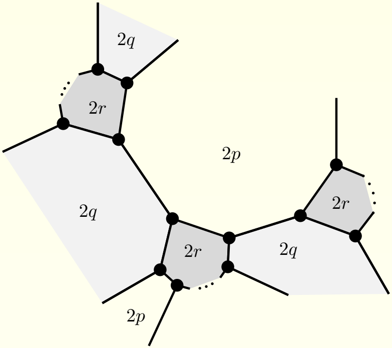

The *Euler characteristic* and the *genus* or *crosscap number* of a hypermap $\mathcal{H}$ is the corresponding quantity of the James representation $J(\mathcal{H})$ of $\mathcal{H}$.

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Each hypermap $\mathcal{H}$ of type (p, q, r) is a quotient of the *universal hypermap* $\mathcal{H}_{(p,q,r)}$ defined as the *full triangle group* $\Delta(p, q, r)$ acting on itself, with

$$\Delta = \Delta(p, q, r) = \langle a,\ b,\ c \mid a^2,\ b^2,\ c^2,\ (bc)^p,\ (ca)^q,\ (ab)^r \rangle\ ,$$

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Similarly, every orientable hypermap $\mathcal{H}^+$ of type (p,q,r) is a quotient of *universal hypermap* $\mathcal{H}^+_{(p,q,r)}$ given by the self-action of the *triangle group*

$$\Delta^+ = \Delta^+(p,q,r) = \langle x,\ y,\ z \mid x^p,\ y^q,\ z^r,\ xyz \rangle\ ,$$

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Method: A combination of branched coverings of surfaces applied to suitable maps (often quite tricky) with loops and multiple edges allowed, enriched by a new technique of constructing new maps from old in the cases when branched covers of the corresponding surfaces do not exists.

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$$\chi = \frac{v}{2}\left(\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_2} - 1\right) ;$$

a type or a triple (k_0, k_1, k_2) will be called *hyperbolic* if $k_0^{-1}+k_1^{-1}+k_2^{-1}<1$.

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$$\text{For } i \in C_3 \text{ define} \quad \nu_i = \mathrm{lcm}\left(\frac{k_{i-1}}{\gcd(k_i, k_{i-1})}, \frac{k_{i+1}}{\gcd(k_i, k_{i+1})}\right) \;;$$

one can show that $\gcd(\nu_0, \nu_1, \nu_2) = 1$ and $k_i\nu_i = \mathrm{lcm}(k_0, k_1, k_2)$, $i \in C_3$.

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The following auxiliary result gives a precise meaning to 'minimality' of the quadruple $(\vartheta; \nu_0, \nu_1, \nu_2)$ with respect to solutions of the system (*).

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Lemma. *For a hyperbolic type (k_0, k_1, k_2) let ν_i for $i \in C_3$ be as defined before. Then, $2k_0\nu_0 = 2k_1\nu_1 = 2k_2\nu_2$, and if ϑ is the common value of the three products, then $\vartheta = 2\,\mathrm{lcm}(k_0, k_1, k_2)$.*

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Moreover, for every solution $(v; n_0, n_1, n_2)$ of the system $(*)$ in positive integers there is an integer t such that $(v; n_0, n_1, n_2) = t(\vartheta; \nu_0, \nu_1, \nu_2)$.

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Similarly, the Euler characteristic of a uniform orientable *hypermap of the same hyperbolic type* κ *is a multiple of the value of* $\chi(\kappa)$ *if* $\chi(\kappa)$ *is even, and a multiple of* $2\chi(\kappa)$ *in the case when* $\chi(\kappa)$ *is odd.*

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Moreover, if $\chi(\kappa)$ is even, the smallest number of vertices in a James representation $J(\mathcal{H})$ of a hypermap $\mathcal{H}$ of type κ, orientable or not, is a multiple of $\vartheta = 2\,\mathrm{lcm}(k_0, k_1, k_2)$.

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Known: *For $n \geq 2$ a surface $\mathcal{S}$ has a smooth n-sheeted cover by a surface with the same orientability characteristic as $\mathcal{S}$. If $\mathcal{S}$ is non-orientable and n is even, then $\mathcal{S}$ also has a smooth n-sheeted orientable cover. In all cases, the n-sheeted covering surface of $\mathcal{S}$ has Euler characteristic $n\chi(\mathcal{S})$.*

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(a) *For* every *hypermap $\mathcal{H}$ of type κ there is an $n \geq 1$ such that $\mathcal{H}$ has $\nu_i n$ faces coloured $i \in C_3$, and $J(\mathcal{H})$ has Euler characteristic $n\chi(\kappa)$.*

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(b) *For each $n \geq 1$ there is a hypermap $\mathcal{H}^{(n)}$ of type κ with $\nu_i n$ i-faces, of Euler characteristic $n\,\chi(\kappa)$, which is an n-fold smooth cover of $\mathcal{H}_{\min}$.*

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Known: *For $n \geq 2$ a surface $\mathcal{S}$ has a smooth n-sheeted cover by a surface with the same orientability characteristic as $\mathcal{S}$. If $\mathcal{S}$ is non-orientable and n is even, then $\mathcal{S}$ also has a smooth n-sheeted orientable cover. In all cases, the n-sheeted covering surface of $\mathcal{S}$ has Euler characteristic $n\chi(\mathcal{S})$.*

Proposition. *Let $\mathcal{H}_{\min}$ be a hypermap of hyperbolic type $\kappa = (k_0, k_1, k_2)$ with ν_i faces coloured $i \in C_3$, with $\vartheta = 2\mathrm{lcm}(k_0, k_1, k_2)$ vertices in $J(\mathcal{H})$, and of Euler characteristic $\chi(\kappa)$. Then:*

(a) *For* every *hypermap $\mathcal{H}$ of type κ there is an $n \geq 1$ such that $\mathcal{H}$ has $\nu_i n$ faces coloured $i \in C_3$, and $J(\mathcal{H})$ has Euler characteristic $n\chi(\kappa)$.*

(b) *For each $n \geq 1$ there is a hypermap $\mathcal{H}^{(n)}$ of type κ with $\nu_i n$ i-faces, of Euler characteristic $n\,\chi(\kappa)$, which is an n-fold smooth cover of $\mathcal{H}_{\min}$. If $\mathcal{H}_{\min}$ is orientable, then so is $\mathcal{H}^{(n)}$; if $\mathcal{H}_{\min}$ is non-orientable, then so is $\mathcal{H}^{(n)}$ if n is odd, but for n even $\mathcal{H}^{(n)}$ can be realised both ways.*

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Proposition. *Let $\kappa' = (k'_0, k'_1, k'_2)$ be a p-reduction of $\kappa = (k_0, k_1, k_2)$.*

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Thus, one has to focus on minimal realisability of *irreducible* triples.

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(b) *If $\mathcal{S}$ is orientable and G has either no pendant vertices, or at least two white pendant vertices but no blue one, or vice versa, or else G has at least two pendant vertices of each of the two colours, then there is an orientable hypermap $\mathcal{H}$ of type (p, q, r) with $2r\ell$ vertices in $J(\mathcal{H})$.*

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After identifying irreducible triples representing types of minimal orientable hypermaps, sorting out those which can be handled by $(p, 1; q, 1)$-graphs and their modifications, and applying non-trivial trickery in the remaining cases (cca 10 pages), we arrived at a solution for orientable hypermaps:

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This gives a complete classification of genera of orientable surfaces which support uniform hypermaps of a given hyperbolic type, and hence also of genera of finite smooth quotients of the corresponding triangle groups.

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By our method of 'doubling', from a given non-orientable hypermap $\mathcal{H}$ with n_i faces of length $2k_i$ ($i \in C_3$) in $J(\mathcal{H})$ one obtains a non-orientable hypermap $\tilde{\mathcal{H}}$ with $2n_i$ faces of length $2k_i$ for $i \in \{0, 1\}$, but with n_2 faces of of length $4k_2$, and such that $\tilde{\mathcal{H}}$ is not a double cover of $\mathcal{H}$.

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The 'doubling' principle for faces of a particular colour $i \in C_3$ in $J(\mathcal{H})$ will be stated in terms of bipartite maps, a special case of which is the *Walsh representation* $W(\mathcal{H})$ of $\mathcal{H}$, obtained from $J(\mathcal{H})$ by contracting all the n_i faces coloured i and then taking the dual of the contracted map.

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Thank you for your (hyper)attention :)