On subvarieties of the moduli space of Riemann surfaces given by group actions

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Sebastián Reyes-Carocca

Departamento de Matemáticas
Facultad de Ciencias
Universidad de Chile
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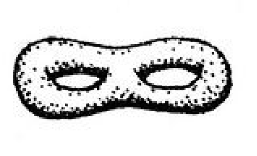
Joint work with Rubén A. Hidalgo, Jennifer Paulhus & Anita M. Rojas

Compact Riemann surfaces

Let $\mathscr{M}_g$ denote the **moduli space** of compact Riemann surfaces of genus $g \geqslant 1$.



Example The case of $g = 1$.

They are complex tori

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}) \text{ where } \tau \in \mathbb{H},$$

and the moduli space $\mathscr{M}_1$ is an **orbit space**

$$\mathbb{H} \to \mathscr{M}_1 = \mathbb{H}/\mathbf{P}\mathrm{SL}(2, \mathbb{Z}) \text{ where } \tau \mapsto [X_\tau]$$

Moduli space of Riemann surfaces

The general situation

Let T_g denote the Teichmüller space of compact Riemann surfaces of genus $g \geqslant 2$.

(a) T_g has structure of a complex analytic manifold of dimension $3g - 3$.

(b) The mapping class group Mod_g of genus g acts on T_g by biholomorphisms

$$\Pi : T_g \mapsto \mathscr{M}_g := T_g/\mathrm{Mod}_g$$

(c) Cartan's theorem: the moduli space $\mathscr{M}_g$ has structure of a normal complex analytic space.

Moduli space of Riemann surfaces

(d) If $t \in T_g$ is represented by the Riemann surface S_t, then

$$\mathrm{Stab}_{\mathrm{Mod}_g}(t) \cong \mathrm{Aut}(S_t).$$

(e) If $g \geqslant 4$ then the singular locus

$$\mathrm{Sing}(\mathscr{M}_g) = \{[S] \in \mathscr{M}_g : \mathrm{Aut}(S) \neq \{1\}\}$$

agrees with the branch locus of the projection

$$T_g \to \mathscr{M}_g.$$

Remark

The singular loci of $\mathscr{M}_2$ and $\mathscr{M}_3$ do not agree with the branch loci.

Equivalences

Assume the genus to be at least two.

Theorem There is an **equivalence** between:

- compact Riemann surfaces,
- (complex projective smooth) algebraic curves,
- orbit spaces $\mathbb{H}/\Gamma$ of the upper-half plane

$$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

by the action of a (co-compact) **Fuchsian group**

$$\Gamma \leqslant \mathrm{Aut}(\mathbb{H}) \cong \mathbb{P}\mathrm{SL}(2, \mathbb{R})$$

Fuchsian groups and uniformization

Riemann's existence Theorem

A finite group G acts on $S \cong \mathbb{H}/\Gamma$ **if and only if** there is a Fuchsian group Δ and a group epimorphism

$$\theta : \Delta \to G \ \text{ such that } \ \ker(\theta) = \Gamma$$

In addition, G is said to act on S with signature $\sigma(\Delta)$ and the Riemann-Hurwitz formula is satisfied

$$2(g-1) = |G|(2h - 2 + \Sigma_{j=1}^{l}(1 - 1/m_j))$$

It is said that the action is **represented** by the **ske** θ

Equivalence of actions

Let S_1 and S_2 be two Riemann surfaces of genus $g \geqslant 2$ endowed with actions

$$\psi_1 : H_1 \to \mathrm{Aut}(S_1) \text{ and } \psi_2 : H_2 \to \mathrm{Aut}(S_2).$$

- ψ_1 and ψ_2 are **topologically equivalent** if there exist a homeomorphism $f : S_1 \to S_2$ and a group isomorphism $\omega : H_1 \to H_2$ such that

$$\psi_1(h) = f\psi_2(\omega(h))f^{-1} \text{ for all } h \in H_1$$

- ψ_1 and ψ_2 are **analytically equivalent** if f is an isomorphism of Riemann surfaces.

Subvarieties of $\mathscr{M}_g$

We denote by

$$\mathscr{M}_g(H, s, \theta) \subset \mathscr{M}_g$$

the locus formed by the points of $\mathscr{M}_g$ representing Riemann surfaces S endowed with the action of a group H and with **topological class** determined by the ske

$$\theta : \Delta \to H$$

where Δ is a Fuchsian group of signature s.

Example $\mathscr{M}_g(\mathbb{Z}_2, s = (0; 2^{2g+2}), \theta)$ = hyperelliptic locus

Subvarieties of $\mathscr{M}_g$

These loci have structure of **closed irreducible subvarieties** of $\mathscr{M}_g$ (Broughton, Harvey-González-Diez)

If $s = (h; m_1, \ldots, m_r)$ then

$$\dim(\mathscr{M}_g(H, s, \theta)) = 3h - 3 + r$$

In general,

$$\mathscr{M}_g(H, s, \theta) \;\text{ is nonsmooth}$$

(= complex d-dimensional family of Riemann surfaces)

Topological versus analytical

Let S' and S'' be two Riemann surfaces of genus $g \geqslant 2$ endowed with actions

$$\psi' : H' \to \mathrm{Aut}(S') \text{ and } \psi'' : H' \to \mathrm{Aut}(S'').$$

Clearly

analytically equivalent $\implies$ topologically equivalent

Question Is the converse true?

Example

We consider the family of Riemann surfaces of genus $g = 5$ with a group of automorphisms isomorphic to

$$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2 = \langle a, b, c : a^8 = b^2 = c^2 = (bc)^2 = 1, bab = a^3, cac = a^5 \rangle$$

acting with signature $(0; 2, 2, 2, 4)$.

- The action is represented by the ske

$$\theta = (b, bc, ab, bca^3)$$

- The surfaces do not have more automorphisms.
- the subgroups

$$H_1 = \langle bc, a \rangle \text{ and } H_2 = \langle ac, b \rangle$$

  are isomorphic to $\mathbb{D}_8$ and nonconjugate.

Example

- We restrict θ to the subgroups H_1 and H_2. The signatures of the induced actions agree, and is $(0; 2^5)$.
- Fact There is a unique topological class of actions of $\mathbb{D}_8$ on genus $g = 5$ with signature $(0; 2^5)$

We conclude that:

the action of H_1 and H_2 are
analytically nonequivalent but topologically equivalent

The normalisation of $\mathscr{M}_g(H, s, \theta)$

We denote by $\{(S', H', \psi')\}$ the analytic equivalence class determined by ψ', and

$$\tilde{\mathscr{M}}_g(H, s, \theta)$$

the set of such classes. The natural correspondence

$$\pi : \tilde{\mathscr{M}}_g(H, s, \theta) \to \mathscr{M}_g(H, s, \theta) \text{ given by } \{(S', H', \psi')\} \mapsto [S']$$

is surjective.

The normalisation of $\mathscr{M}_g(H, s, \theta)$

Theorem (Harvey & González-Diez, 92′)

- $\tilde{\mathscr{M}}_g(H, s, \theta)$ has structure of a normal complex analytic space (it is a moduli space itself),
- the map

$$\pi : \tilde{\mathscr{M}}_g(H, s, \theta) \to \mathscr{M}_g(H, s, \theta)$$

  turns out to be a morphism of analytic complex spaces, and
- the surjection π is the **normalisation** of $\mathscr{M}_g(H, s, \theta)$.

The normalisation of $\mathscr{M}_g(H,s,\theta)$

In other words, the following statements are **equivalent**.

- $\mathscr{M}_g(H,s,\theta)$ is non-normal subvariety of $\mathscr{M}_g$.
- There is a Riemann surface

$$[X_0] \in \mathscr{M}_g(H,s,\theta)$$

endowed with the action of a group H' that is **topologically but not analytically equivalent** to the action of H determined by θ.

What does normality mean?

The concept of **normal variety** was introduced by Zariski in 1939. Following Mumford, normality can be understood as

a way to separate the "branches" of an algebraic variety at a singular point.

The importance of normal varieties lie in part in the fact that **the co-dimension of their singular loci is** $\geqslant 2$.

- normal complex curves are smooth, and
- the singularities of normal complex surfaces are isolated.

Some known examples

Example 1 (González-Diez, 95′)

If p is a prime number then

$$\mathscr{M}_g(\mathbb{Z}_p, s = (0; p, \overset{r}{\ldots}, p), \theta)$$

is **normal**, for each action θ. Here $g = (r-2)(p-1)/2$.

In other words, for cyclic groups acting with genus zero

topological action $\iff$ analytic action

Some known examples

Example 1 (González-Diez & Hidalgo, 97′)

The first known example of a **non-normal** subvariety associated to regular covers of the projective line $\mathbb{P}^1$ is the two-dimensional subvariety of $\mathscr{M}_9$ given by

$$\mathscr{M}_9(\mathbb{Z}_8, s = (0; 4, 4, 4, 8, 8), \theta),$$

for some action θ suitably chosen. This subvariety has a one-dimensional sublocus of non-normal points.

Generalisation (Carvacho, 13′)

$\mathscr{M}_{3(2^n-1)}(\mathbb{Z}_{2^{n+1}}, s = (0; 2^n, 2^n, 2^n, 2^{n+1}, 2^{n+1}), \theta_n)$ where $n \geqslant 2$.

Some known examples

Example 3 (Cirre, 04′)

The first example of a non-normal subvariety associated to regular **non-cyclic** covers of $\mathbb{P}^1$ is the three-dimensional subvariety

$$\mathscr{M}_3(\mathbb{Z}_2^2, s = (0; 2, 2, 2, 2, 2, 2), \theta_h),$$

where θ_h stands for ske corresponding to surfaces that lie in the hyperelliptic locus.

More examples : $H \cong \mathbb{Z}_4$ in genus 3 and $H \cong \mathbb{Z}_8$ in genus 9.

Remark It is not difficult to construct examples of non-normal subvarieties if the signature of the action is a non-genus-zero action.

More examples (for genus-zero actions)?

Question & Problems

- Find the complete set of non-normal subvarieties of $\mathscr{M}_g$ for g small, and describe their non-normal points.
- Once g is fixed, does $\mathscr{M}_g$ contain a non-normal subvariety of the desired form (=genus zero action)?
- More generally, does each $\mathscr{M}_g$ contain a non-normal subvariety of the resided form?
- If so, which is the “biggest” non-normal subvariety?
- Once a class $\mathscr{G}$ of groups is fixed, does each $G \in \mathscr{G}$ yields a non-normal subvariety of $\mathscr{M}_g$ for some g?

Some results

Theorem

There is exactly **one** non-normal irreducible subvariety of $\mathscr{M}_2$ of the form $\mathscr{M}_2(H, s, \theta)$.

This subvariety has dimension two and is formed by the surfaces with a group of automorphisms isomorphic to

$$H = \mathbb{Z}_2^2 \text{ acting with signature } s = (0; 2^5).$$

In addition, the set of non-normal points is formed by the surfaces with a group of automorphisms isomorphic to

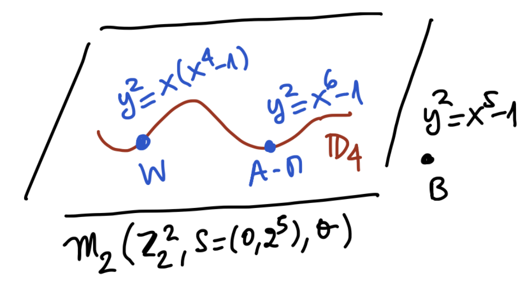
$$\mathbf{D}_4 \text{ acting with signature } (0; 2^3, 4).$$

Some results

In particular, the Accola-Maclachlan curve and the Wiman curve of type II

$$y^2 = x^6 - 1 \;\; \text{and} \;\; y^2 = x(x^4 - 1)$$

respectively, are non-normal points of it.



Some results

Theorem

There are exactly **seven** non-normal irreducible subvarieties of $\mathscr{M}_3$ of the form $\mathscr{M}_3(H, s, \theta)$.

- As in the case of genus $g = 2$ we can describe explicitly the full set of non-normal points, in each case.
- Cirre's example is recovered.
- the groups appearing are: $\mathbf{S}_4$, $\mathbb{Z}_2^3$, $\mathbf{D}_4$, $\mathbb{Z}_4$, $\mathbb{Z}_2^2$ and $\mathbb{Z}_2$

Some results

- dimensions $1, 2, 3$ and 4 (Wiman, Fermat, Klein)
- Problem in higher genus: several groups, topological classes, full automorphism groups, etc

Some results

Theorem

For each integer $g \geqslant 3$, the g-dimensional subvariety of $\mathscr{M}_g$

$$\mathscr{M}_g(\mathbb{Z}_2^2, s = (0; 2^{g+3}), \theta_h)$$

is non-normal, where θ_h stands for hyperelliptic stratum. The subvariety is formed by the curves

$$y^2 = \Pi_{i=1}^{g+1}(x - a_i)(x - \frac{1}{a_i})$$

where $a_1, \ldots, a_{g+1}$ are pairwise distinct complex numbers satisfying that $a_i a_j \neq 1$ for all $i, j \in \{1, \ldots, g+1\}$.

Some results

The group of automorphisms of S that is isomorphic to $\mathbb{Z}_2^2$ is represented by

$$\langle (x,y) \mapsto (x,-y),\ (x,y) \mapsto (1/x, y/x^{g+1}) \rangle.$$

A set of **non-normal points** is given as follows.

a. If g is odd: the Riemann surfaces with a group of automorphisms isomorphic to

$$\mathbb{Z}_2 \times \mathbf{D}_{g+1} \text{ acting with signature } (0; 2^3, g+1),$$

and represented by the algebraic curves

$$y^2 = (x^{g+1} - a)(x^{g+1} - \tfrac{1}{a}) \text{ where } a \neq 0, \pm 1.$$

b. If g is even: similar!

Some results

Corollary

Each moduli space $\mathscr{M}_g$ has a non-normal subvariety of the desired type.

More precisely:

For each $g \geqslant 2$, there are infinitely many compact Riemann surfaces of genus g **that are branched regular covers of $\mathbb{P}^1$** endowed with two groups of automorphisms that are topologically but not conformally conjugate.

Some results

Theorem

For each odd integer $g \geqslant 3$, the subvariety

$$\mathscr{M}_g(\mathbf{D}_{2(g-1)}, s = (0; 2^5), \theta)$$

is non-normal, where θ stands for the ske representing the unique action of $\mathbf{D}_{2(g-1)}$ on genus g with signature s. A set of non-normal points is formed by the Riemann surfaces with a group of automorphisms isomorphic to

$$\mathbb{Z}_{2(g-1)} \rtimes \mathbb{Z}_2^2 \text{ acting with signature } (0; 2^3, 4).$$

(the case $g = 5$ was discussed before)

The novelty: First non-abelian covers of $\mathbb{P}^1$

Some results

Theorem

For each odd integer $n \geqslant 3$, the moduli space $\mathscr{M}_{(n-1)^2}$ contains a non-normal subvariety of dimension $n-1$ of the form

$$\mathscr{M}_{(n-1)^2}(\mathbb{Z}_{2n}, s = (0; 2^2, n, \overset{n}{\ldots}, n), \theta)$$

A set of non-normal points is formed by the Riemann surfaces with a group of automorphisms isomorphic to

$\mathbb{Z}_2 \times \mathbb{Z}_{2n}$ acting with signature $(0; 2^2, n, \overset{(n-1)/2}{\ldots}, n, 2n)$.

This result extends Carvacho and Hidalgo-GonzálezDiez cases.

Thanks!

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