

# Clifford bundles over Klein surfaces

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## Klein surfaces

A **Klein surface** is a compact topological surface  $Y$  equipped with a dianalytic structure.

A Klein surface  $Y$  with topological genus  $g$  and  $q$  boundary components is denoted by  $Y_{\alpha,p,q}$  where  $p = \alpha g$  for  $\alpha = \begin{cases} 2 & \text{if } Y \text{ orientable} \\ 1, & \text{if } Y \text{ nonorientable.} \end{cases}$

The integer  $d = p + q - 1$  is the **algebraic genus** of  $Y$ .

**Non-euclidean crystallographic (NEC) group** is a discrete in the topology of  $\mathbb{R}^4$  subgroup of the group of isometries of the hyperbolic plane  $\mathcal{H}$  with compact orbit space.

A **surface NEC group** has no conformal elements with fixed points.

**Thm(Preston):** If  $d \geq 2$ , then  $Y_{\alpha,p,q} \simeq \mathcal{H}/\Lambda$  for a surface NEC group  $\Lambda$ .

## NEC group uniformizing $Y_{\alpha,p,q}$

$\Lambda_{\alpha,p,q}$  denotes an NEC group with the signature

$$\sigma(\Lambda_{\alpha,p,q}) = \left(\frac{p}{\alpha}; \pm; [-]; \{(-), \dots, (-)\}\right), \quad (1)$$

If  $\alpha = 2$ , then  $p$  is even, the sign is  $+$  and  $\Lambda_{\alpha,p,q}$  is generated by: hyperbolic elements  $a_i, b_i$  (for  $i = 1, \dots, \frac{p}{2}$ ), reflections  $c_j$  and connecting generators  $e_j$  (for  $j = 1, \dots, q$ ) such that

$$c_j^2 = 1, [e_j, c_j] = 1$$

and

$$e_1 \cdots e_q [a_1, b_1] \cdots [a_{\frac{p}{2}}, b_{\frac{p}{2}}] = 1. \quad (2)$$

If  $\alpha = 1$ , then the sign is  $-$  and the generators  $a_i, b_i$  are replaced with glide reflections  $d_1, \dots, d_p$ , and the relation (2) is replaced with

$$e_1 \cdots e_q d_1^2 \cdots d_p^2 = 1. \quad (3)$$

Any generators of  $\Lambda_{\alpha,p,q}$  that satisfy the above relations are called the **canonical generators**.

## Clifford algebra $Cl_{r,s}$

$V = \mathbb{R}^{r,s}$ -a real vector space with a basis  $\{v_1, \dots, v_n\}$  for  $n = r + s$ ,

$Q_{r,s} : V \rightarrow \mathbb{R}$ -a non-degenerate quadratic form

$$Q_{r,s}(v_k) = \begin{cases} 1 & \text{if } 1 \leq k \leq r \\ -1, & r + 1 \leq k \leq n. \end{cases}$$

The **Clifford algebra** is a real associative algebra  $Cl_{r,s}$  with a linear map  $i : V \rightarrow Cl_{r,s}$ , called a **Clifford map**, such that

$$(1) \quad i(v)^2 = -Q_{r,s}(v)1 \text{ for all } v \in V,$$

(2)  $Cl_{r,s}$  is **universal** which means that for any real associative algebra  $C$  with unit  $1_C$  and a Clifford map  $F : V \rightarrow C$ , there exists a unique algebra homomorphism  $\tilde{F} : Cl(r, s) \rightarrow C$  such that  $\tilde{F} \circ i = F$ .

## Even Clifford subalgebra $\text{Cl}_{r,s}^+$

The **orthogonal group**:

$$O(r, s) = \{f \in \text{End}(V) : Q_{r,s} \circ f = Q_{r,s}\}$$

If  $f \in O(r, s)$ , then  $\det f = \pm 1$ .

The **special orthogonal group**:

$$SO(r, s) = \{f \in O(r, s) : \det f = 1.\}$$

By the universality of  $\text{Cl}_{r,s}$ , for any  $f \in O(r, s)$ , there is an automorphism  $\varphi_f$  of  $\text{Cl}_{r,s}$ .

The subalgebra  $\text{Cl}_{r,s}^+$  consists of all elements in  $\text{Cl}_{r,s}$  that are fixed by the automorphism  $\varphi_f$  induced by an orthogonal map  $f$  defined by

$$f(v) = -v \text{ for } v \in V.$$

## Pin and Spin groups

**Pin group** is a multiplicative subgroup  $\text{Pin}(r, s)$  of  $\text{Cl}_{r,s}$  generated by all  $v \in V$  with  $Q_{r,s}(v) = \pm 1$ .

**Spin group** is  $\text{Spin}(r, s) = \text{Pin}(r, s) \cap \text{Cl}_{r,s}^+$ .

A subgroup  $M_{r,s}$  of  $\text{Pin}(r, s)$  generated by the canonical basis  $v_1, \dots, v_n$  of  $V = \mathbb{R}^{r,s}$  is called **base group**.

$M_{r,s}^+ = M_{r,s} \cap \text{Cl}_{r,s}^+$  is a subgroup of  $\text{Spin}(r, s)$ .

Relations in  $M_{r,s}$ :

$$v_k^2 = \begin{cases} 1 & \text{if } 1 \leq k \leq r \\ -1, & r+1 \leq k \leq n \end{cases} \quad v_k v_l = -v_l v_k \text{ for } k \neq l.$$

$|M_{r,s}| = 2^{n+1}$  and  $|M_{r,s}^+| = 2^n$  for  $n > 1$ .

We denote  $v_1, \dots, v_n$  by  $w_1, \dots, w_r, \varepsilon_1, \dots, \varepsilon_s$ .

## Clifford actions defining Klein surfaces

Let  $1 \rightarrow \Gamma \rightarrow \Lambda \xrightarrow{\theta} M_{r,s} \rightarrow 1$  for  $\Lambda = \Lambda_{\alpha,p,q}$ .  
Then  $M_{r,s} \simeq \Lambda/\Gamma$  acts on  $X = \mathcal{H}/\Gamma$  by

$$\lambda\Gamma \cdot \Gamma z = \Gamma\lambda(z) \text{ for } z \in \mathcal{H} \text{ and } \lambda \in \Lambda$$

and  $X/M_{r,s} \simeq (\mathcal{H}/\Gamma)/(\Lambda/\Gamma) \simeq \mathcal{H}/\Lambda \simeq Y_{\alpha,p,q}$ .

We say that that  $(\Lambda, \theta, M_{r,s})$  is a **Clifford action defining the Klein surface**  $Y_{\alpha,p,q}$ .

The surface  $Y' \simeq \mathcal{H}/\Lambda'$  for  $\Lambda' = \theta^{-1}(M_{r,s}^+)$  is called the **Clifford double cover** of  $Y$  induced by the action  $(\Lambda, \theta, M_{r,s})$ .

**Thm.** Any Klein surface  $Y = Y_{\alpha,p,q}$  is defined by a Clifford action for which the induced Clifford double cover  $Y'$  is the canonical double cover  $Y^+$  of  $Y$ .

## The standard Clifford action

$\theta : \Lambda_{\alpha.p.q} \rightarrow M_{p,q}$  defined by:

$$\begin{aligned}
 \theta(c_j) &= \varepsilon_j, & j &= 1, \dots, q, \\
 \theta(e_q) &= w_p^{2^\alpha}, \theta(e_j) = 1 & j &= 1, \dots, q-1, \\
 \alpha &= 2 : \\
 \theta(a_i) &= w_{2i-1}, & i &= 1, \dots, \frac{p}{2}, \\
 \theta(b_i) &= w_{2i}, & i &= 1, \dots, \frac{p}{2} - 1, \\
 \theta(b_{\frac{p}{2}}) &= \begin{cases} w_1 w_p & \text{if } \frac{p}{2} \equiv 1 \pmod{2}, q = 0 \\ w_p, & \text{otherwise,} \end{cases} \\
 \alpha &= 1 : \\
 \theta(d_i) &= w_i, & i &= 1, \dots, p-1, \\
 \theta(d_p) &= \begin{cases} w_{p-2} w_{p-1} w_p & \text{if } p \equiv 1 \pmod{2}, q = 0 \\ w_p, & \text{otherwise.} \end{cases}
 \end{aligned} \tag{4}$$

is called the **standard epimorphism**.

$A_{\alpha,p,q} = (\Lambda, \theta, M_{p,q})$  is the **standard Clifford action** defining  $Y = Y_{\alpha,p,q}$ .

$Y' \simeq \mathcal{H}/\Lambda'$  for  $\Lambda' = \theta^{-1}(M_{r,s}^+)$  is the **standard Clifford double cover** of  $Y$ .

## Principal Pin-bundles over Klein surfaces

Let  $A = (\Lambda, \theta, M_{r,s})$  be a Clifford action defining a Klein surface  $Y = \mathcal{H}/\Lambda$ .

Let  $\mathcal{H}_\theta$  be the orbit space of  $\mathcal{H} \times \text{Pin}(r, s)$  under the action of  $\Lambda$ , where

$$\lambda \cdot (z, g) = (\lambda(z), g\theta(\lambda)^{-1}) \text{ for } (z, g) \in \mathcal{H} \times \text{Pin}(r, s).$$

A mapping  $p_\theta : \mathcal{H}_\theta \rightarrow Y$  defined by

$$p_\theta(\langle z, g \rangle) = \Lambda z$$

is the **principle**  $\text{Pin}(r, s)$ -**bundle**, where  $\text{Pin}(r, s)$  acts on  $\mathcal{H}_\theta$  by  $g_1 \cdot \langle z, g_2 \rangle = \langle z, g_1 g_2 \rangle$ . We can also construct the **principal Spin-bundle over**  $Y'$ .

For representations  $\rho : \text{Pin}(r, s) \rightarrow \text{Gl}(W)$  and  $\rho' : \text{Spin}(r, s) \rightarrow \text{Gl}(W')$ , there are associated vector bundles

$$\mathcal{H} \times_\Lambda W \rightarrow Y \text{ and } \mathcal{H} \times_{\Lambda'} W' \rightarrow Y'$$

## Clifford bundles over Klein surfaces

**Vector representations** of groups  $\text{Pin}(r, s)$  and  $\text{Spin}(r, s)$  are homomorphisms:

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Pin}(r, s) \xrightarrow{\rho_{r,s}} O(r, s) \rightarrow 1$$

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Spin}(r, s) \xrightarrow{\rho_{r,s}} SO(r, s) \rightarrow 1.$$

Assume  $A = (\Lambda_{\alpha, r, s}, \theta, M_{r, s})$  defines  $Y = Y_{\alpha, r, s}$ .  
 $\Lambda$  acts on  $\mathcal{H} \times \text{Cl}_{r, s}$  by

$$\lambda \cdot (z, c) = (\lambda(z), \varphi_\lambda(c)) \text{ for } z \in \mathcal{H}, c \in \text{Cl}_{r, s},$$

where  $\varphi_\lambda \in \text{Aut}(\text{Cl}_{r, s})$  is induced by  $\rho_{r, s}(\theta(\lambda))$ .

Let  $\mathcal{H}_{r, s} = (\mathcal{H} \times \text{Cl}_{r, s})/\Lambda$  and let  $\langle z, c \rangle$  denote the orbit of a point  $(z, c)$ . The surjection

$$\pi_{r, s} : \mathcal{H}_{r, s} \rightarrow Y, \pi_{r, s}(\langle z, c \rangle) = \Lambda z$$

is a **Clifford bundle** over  $Y$  induced by  $A$ .  
There is also a Clifford bundle over  $Y'$  with typical fibre  $\text{Cl}_{r, s}^+$ .

**Thm E.T.** Let  $\tilde{f} : Cl_{r,s} \rightarrow Cl_{r_1,s_1} \otimes Cl_{r_2,s_2}$  be an isomorphism:

- (a)  $Cl_{r'+1,s'+1} \simeq Cl_{r',s'} \otimes Cl_{1,1}$ ,
- (b)  $Cl_{0,s'+2} \simeq Cl_{s',0} \otimes Cl_{0,2}$
- (c)  $Cl_{r'+2,0} \simeq Cl_{0,r'} \otimes Cl_{2,0}$ ,
- (d)  $Cl_{r'+2l,s'+2l} \simeq Cl_{r',s'} \otimes Cl_{2l,2l}$ .

Then  $Y_{\alpha,r,s} = \mathcal{H}/\Lambda$  is definable by Clifford actions  $A = (\Lambda, \theta, M_{r,s})$  and  $A_i = (\Lambda, \theta_i, M_{r_i,s_i})$   $i = 1, 2$  such that vector bundle  $\mathcal{H} \times_{\Lambda} \mathbb{R}^{r,s} \rightarrow Y$  induced by  $A$  associated with vector representation of the group  $\text{Pin}(r,s)$  is equivalent to the tensor product of vector bundles

$$\mathcal{H} \times_{\Lambda} \mathbb{R}^{r_i,s_i} \rightarrow Y, \quad i = 1, 2$$

induced by  $A_1$  and  $A_2$  associated with vector representations of  $\text{Pin}(r_i, s_i)$ ,  $i = 1, 2$ .

Clifford bundle over  $Y$  induced by  $A$  is equivalent to tensor product of Clifford bundles induced by  $A_1$  and  $A_2$ .

## The Whitney sum of vector bundles

**Thm E.T.** Let  $A_{\alpha, p_l, q_l} = (\Lambda_l, \theta_l, M_{p_l, q_l})$  be a standard Clifford action and let  $B_l$  be the set of canonical generators of  $\Lambda_l = \Lambda_{\alpha, p_l, q_l}$  for  $l = 1, 2$ .

Then  $B_1 \cup B_2$  generates an NEC group

$$\Lambda \simeq \Lambda_{\alpha, p, q} \text{ for } p = p_1 + p_2, \quad q = q_1 + q_2$$

and the standard vector bundle

$$\mathcal{H} \times_{\Lambda} \mathbb{R}^{p, q} \rightarrow Y_{\alpha, p, q}$$

associated with a vector representation of the group  $\text{Pin}(p, q)$  is equivalent to the Whitney sum of vector bundles  $\mathcal{H} \times_{\Lambda} \mathbb{R}^{p_l, q_l} \rightarrow Y$  for  $l = 1, 2$  associated with representations  $\rho_l$  induced by

$$\rho_l(\lambda) = \begin{cases} \rho_{p_l, q_l} \circ \theta_l(\lambda) & \text{if } \lambda \in B_l \\ \text{id}_{\mathbb{R}^{p_l, q_l}}, & \text{if } \lambda \notin B_l, \end{cases} \quad (5)$$

where  $\rho_{p_l, q_l} : \text{Pin}_{p_l, q_l} \rightarrow O(p_l, q_l)$  denotes vector representation of the Pin group.

## Literature

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