A group of mapping classes and its action on closed Riemann surfaces of genus two

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Aim of this talk

In this talk we introduce a group G of mapping classes and examine its action on the Teichmüller space $\mathcal{T}_2$ of closed Riemann surfaces of genus two. The group G is defined by changing the markings on surfaces through cutting and pasting fundamental polygons and rotating them. We define $\mathcal{T}_2$ as a subset of $\mathbb{R}^7$ and express the generators of G in terms of seven variables. Finally, we demonstrate how G acts on marked extremal Riemann surfaces.

$$\begin{array}{ccc} \mathcal{T}_2 & \xrightarrow{g \in G} & \mathcal{T}_2 \\ \Phi \downarrow & & \downarrow \Phi \\ \mathbb{R}^7 & \longrightarrow & \mathbb{R}^7 \end{array}$$

Teichüller space of genus g

$\mathcal{T}_g$: Teichüller space of genus $g \geq 2$
= the space of all marked closed Riemann surfaces of genus g
= the space of all canonical polygons[1]

Canonical polygon

We call a hyperbolic polygon P of $4g$ sides a canonical polygon if

1. P is convex.
2. each pair of opposite sides of P has the same length.
3. P satisfies some angle conditions.

Identifying each pair of opposite sides of P we have a closed Riemann surface of genus g. Then the surface inherits a marking induced by the sides of P.

[1] P. Schmutz Schaller, Teichmüller space and fundamental domains of Fuchsian groups, Enseign. Math. **45** (1999) no. 2, 169–187.

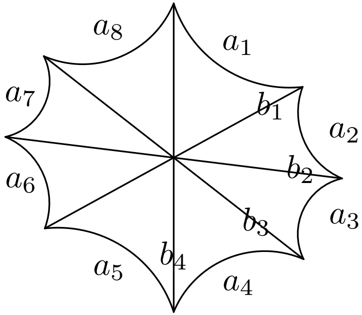
Mapping class groups

Let $\mathcal{M}_g$ be the moduli space of closed Riemann surfaces of genus g, namely, the set of all biholomorphic equivalence classes of such surfaces. The mapping class group $\mathcal{MCG}_g$ is the group acting on $\mathcal{T}_g$ which "changes the markings".

$$\mathcal{M}_g = \mathcal{T}_g/\mathcal{MCG}_g.$$

Teichmüller space of genus two

The 7 length a_1, a_2, a_3, b_1, b_2, b_3, b_4 of canonical polygon give an embedding of $\mathcal{T}_2$ into $\mathbb{R}^7$. We take $(x_1, \ldots, x_7)$ as a system of global coordinates of $\mathcal{T}_2$.



$$x_k := \cosh a_k, \ k = 1, 2, 3$$

$$x_{3+k} := \cosh \frac{b_k}{2}, \ k = 1, 2, 3, 4$$

Theorem 2.1 (N, 2012)

Let $\Phi : \mathcal{T}_2 \to \mathbb{R}^7$; $P \mapsto (x_1, \ldots, x_7)$. *Then* $(x_1, \ldots, x_7)$ *satisfies the following conditions:*

- (i) *(i)* $|X| < 1$, $|Y| < 1$, $|Z| < 1$;
- (ii) *(ii)* $X + Y + Z - 1 > \sqrt{2(1-X)(1-Y)(1-Z)}$;
- (iii) *(iii)* $A^2 + B^2 + C^2 + D^2 + 2ABCD - 2 = 2\sqrt{(1-A^2)(1-B^2)(1-C^2)(1-D^2)}$;

where

$$X := (x_4x_7 - x_1)/\sqrt{(x_4^2-1)(x_7^2-1)},$$
$$Y := (x_4x_5 - x_2)/\sqrt{(x_4^2-1)(x_5^2-1)},$$
$$Z := (x_5x_6 - x_3)/\sqrt{(x_5^2-1)(x_6^2-1)},$$

$$A := (x_1 + x_4 + x_7 + 1)/\sqrt{2(x_1+1)(x_4+1)(x_7+1)},$$
$$B := (x_2 + x_4 + x_5 + 1)/\sqrt{2(x_2+1)(x_4+1)(x_5+1)},$$
$$C := (x_3 + x_5 + x_6 + 1)/\sqrt{2(x_3+1)(x_5+1)(x_6+1)},$$
$$D := (x_6 + x_7 + x_8 + 1)/\sqrt{2(x_6+1)(x_7+1)(x_8+1)},$$

$$x_8 :=$$
$$x_6x_7 + \sqrt{(x_6^2-1)(x_7^2-1)}\Big(XYZ - X\sqrt{(1-Y^2)(1-Z^2)} - Y\sqrt{(1-Z^2)(1-X^2)} - Z\sqrt{(1-X^2)(1-Y^2)}\Big).$$

Examples of mapping classes

We give three examples of mapping classes by using "cut and paste".

ρ: a periodic element

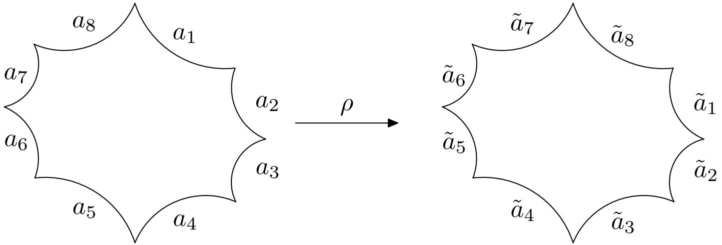
We define an periodic element $\rho \in \mathcal{MCG}_2$. For $P \in \mathcal{T}_2$,

$$\rho : P \mapsto \tilde{P},$$

where $\tilde{P}$ is the polygon with sides $\tilde{a}_j = a_{j+1}$ for every j. $(\tilde{a}_8 = a_1)$.

Note:

1. ρ is a periodic element of order 4.
2. ρ fixes the regular octagon.


a_8
a_1
a_7
a_2
a_6
a_3
a_5
a_4
ρ
$\tilde{a}_7$
$\tilde{a}_8$
$\tilde{a}_6$
$\tilde{a}_1$
$\tilde{a}_5$
$\tilde{a}_2$
$\tilde{a}_4$
$\tilde{a}_3$

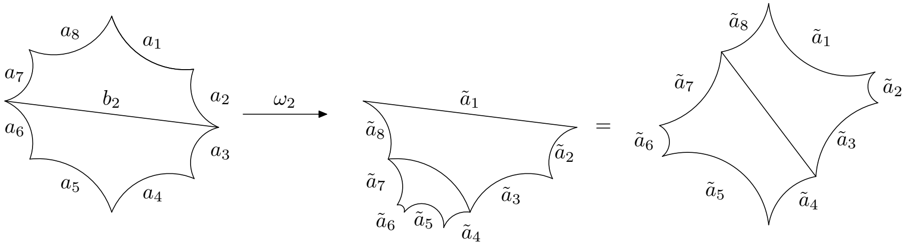
ω_{12}

Next we define $\omega_{12} \in \mathcal{MCG}_2$. For $P \in \mathcal{T}_2$, let $\tilde{P}$ be a polygon obtained by cutting P along b_2 into two pieces and pasting them along sides a_1 and a_5.

$$\omega_{12} : P \mapsto \tilde{P},$$

We label sides $\tilde{a}_1, \ldots, \tilde{a}_8$ as
$\tilde{a}_1 = b_2$, $\tilde{a}_2 = a_3$, $\tilde{a}_3 = a_4$, $\tilde{a}_4 = f(a_2)$
$\tilde{a}_5 = f(b_2)$, $\tilde{a}_6 = f(a_7)$, $\tilde{a}_7 = f(a_8)$, $\tilde{a}_8 = a_6$,
where f denotes the side-pairing mapping which maps a_1 onto a_5.

Note: $\tilde{P}$ is also a canonical polygon.



a_8
a_1
a_7
b_2
a_2
a_6
a_3
a_5
a_4
ω_2
$\tilde{a}_1$
$\tilde{a}_8$
$\tilde{a}_2$
$\tilde{a}_7$
$\tilde{a}_3$
$\tilde{a}_6$
$\tilde{a}_5$
$\tilde{a}_4$
$=$
$\tilde{a}_8$
$\tilde{a}_1$
$\tilde{a}_7$
$\tilde{a}_2$
$\tilde{a}_6$
$\tilde{a}_3$
$\tilde{a}_5$
$\tilde{a}_4$

We define ω_{ij} in a similar way as ω_{12}.

For $P \in \mathcal{T}_2$, cup P along b_j into two pieces and paste them along a_i and a_{i+4}. The sides of the resulting polygon $\tilde{P}$ are labelled as $\tilde{a}_1, \ldots, \tilde{a}_8$ clockwise, where $\tilde{a}_1 = b_j$.

When $i = 1$, we denote $\omega_{1,j}$ briefly by ω_j.

Proposition 3.1

Let G *be the subgroup of* $\mathcal{MCG}_2$ *generated by all* ω_{ij}*. Then* $G = \langle \rho, \omega_2 \rangle$*.*

Proposition 3.2

The following relations hold:

1. $\omega_1 = \omega_2^3$, $\omega_3 = \omega_2^{-1}$, $\omega_4 = \omega_2^{-3}$.
2. $\omega_{ij} = \omega_{j-i+1}\rho^{i-1}$
3. $\rho^2\omega_2^{-3}\rho^2 = \omega_2\rho\omega_2$.

The subscripts are integers modulo 4.

For $P \in \mathcal{MCG}_2$ with its coordinates $\Phi(P) = (x_1, \ldots, x_7)$,

$$\begin{aligned}\Phi(\tilde{P}) &= \Phi\omega_2\Phi^{-1}(x_1, \ldots, x_7)\\ &= \left(2x_5^2 - 1, x_3, x_8, \tilde{x}_4, \tilde{x}_5, \sqrt{\frac{x_1+1}{2}}, \tilde{x}_7\right),\end{aligned}$$

where

$$\tilde{x}_4 = \frac{x_5(2x_4+x_7)-x_2+\sqrt{(x_5^2-1)(x_7^2-1)}(XY-\sqrt{(1-X^2)(1-Y^2)})}{\sqrt{2(x_1+1)}},$$

$$\tilde{x}_5 = \frac{x_4x_6+x_8+\sqrt{(x_4^2-1)(x_6^2-1)}(YZ-\sqrt{(1-Y^2)(1-Z^2)})}{\sqrt{2(x_1+1)}},$$

$$\tilde{x}_7 = \frac{x_2+x_5x_7-\sqrt{(x_5^2-1)(x_7^2-1)}(XY-\sqrt{(1-X^2)(1-Y^2)})}{\sqrt{2(x_1+1)}}.$$

Theorem 1

Let $P_0 \in \mathcal{T}_2$ *be the regular octagon, so that its coordinate is*

$$\Phi(P_0) = (a, a, a, b, b, b, b),$$

where $a = 5 + 4\sqrt{2}$ *and* $b = 3 + 2\sqrt{2}$.
Then the coordinates of $\omega_2^n(P_0)$ *are in* $\mathbb{Z}[\sqrt{2}]^7$ *for all* $n \in \mathbb{Z}$.

Extremal surfaces

Definition 4.1

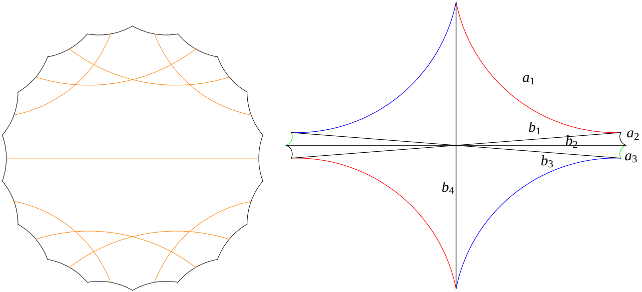
A closed Riemann surface of genus $g \geq 2$ is said to be extremal if it admits a hyperbolic disk with radius r_g, where

$$\cosh r_g = \frac{1}{2\sin(\pi/(12g-6))}.$$

This is a surface which attains the maximum of the injectivity radius function on $\mathcal{M}_g$. There are 9 extremal surfaces in $\mathcal{M}_2$.

We consider the images of extremal surfaces by ρ and ω_2.

Let S_1 be an extremal surface obtained from side-pairings of the regular 18-gon in the left-hand figure. One of the canonical polygons P_1 for S_1 is depicted as the right-hand figure.



The coordinate of P_1 is approximately as follows:

$$\Phi(P_1) = (112.2, 5.851, 5.851, 22.43, 63.33, 22.43, 5.411).$$

$$\Phi(P_1) \overset{\omega_2}{\mapsto} \begin{bmatrix} \mathbf{8019.6} \\ \mathbf{5.851} \\ \mathbf{112.2} \\ \mathbf{211.2} \\ \mathbf{73.89} \\ \mathbf{7.522} \\ \mathbf{23.17} \end{bmatrix} \overset{\omega_2}{\mapsto} \begin{bmatrix} \mathbf{10918.3} \\ \mathbf{112.2} \\ \mathbf{5.851} \\ \mathbf{251.1} \\ \mathbf{22.43} \\ \mathbf{63.33} \\ \mathbf{22.43} \end{bmatrix} \overset{\omega_2}{\mapsto} \begin{bmatrix} \mathbf{1005.5} \\ \mathbf{5.851} \\ \mathbf{5.851} \\ \mathbf{75.52} \\ \mathbf{211.2} \\ \mathbf{73.89} \\ \mathbf{7.522} \end{bmatrix} \overset{\omega_2}{\mapsto} \cdots .$$

We introduce a distance on $\mathcal{T}_2$:

Definition 4.2

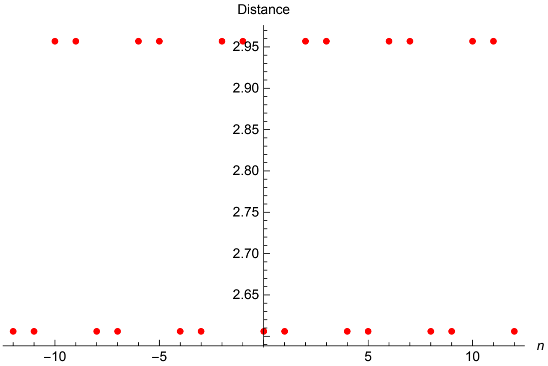
For $\Phi(P) = (x_1, \dots, x_7)$*,* $\Phi(P') = (x'_1, \dots, x'_7)$*,*

$$\mathrm{dist}(P, P') := \sum_{k=1}^{7} \left| \log \frac{\cosh^{-1}(x'_k)}{\cosh^{-1}(x_k)} \right|.$$

$$(\cosh^{-1}(x_1), \dots, \cosh^{-1}(x_7)) \\ = (a_1, a_2, a_3, b_1/2, b_2/2, b_3/2, b_4/2).$$

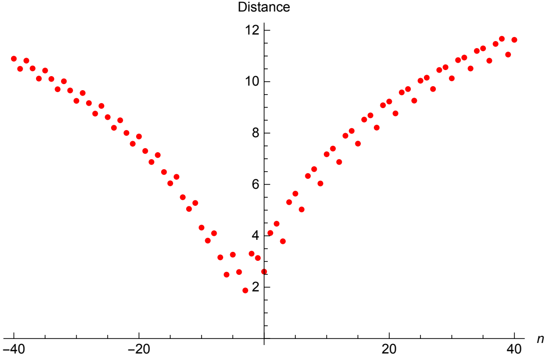
Let $P_0 \in \mathcal{T}_2$ be the regular octagon. We consider it as the basepoint of $\mathcal{T}_2$.

$\mathrm{dist}(P_0, \rho^n(P_1))$

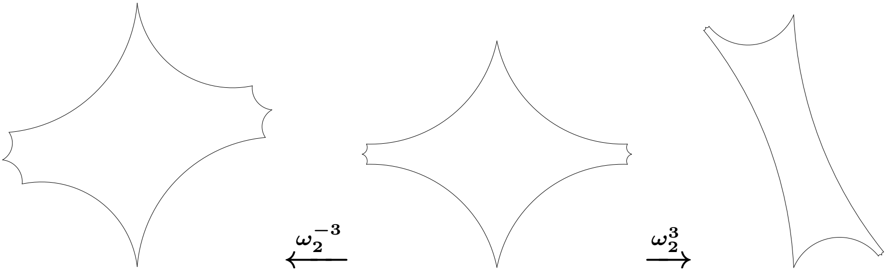


Minimum: 2.606.

$\mathrm{dist}(P_0, \omega_2^n(P_1))$



Minimum: 1.874.


ω_2^{-3}
ω_2^{3}

Calculating distances between P_0 and the other 9 extremal surfaces, we have

Theorem 4.3

Let $B_{P_0}(r) = \{P \in \mathcal{T}_2 \mid \mathrm{dist}(P, P_0) < r\}$ *and let* $G(P) = \{g(P) \mid g \in G\}$ *be the orbit of* $P \in \mathcal{T}_2$. *For the marked extremal surfaces* P_i, $i = 1, 2, \ldots, 9$, *we deduce that*

$$B_{P_0}(r) \cap G(P_i) \neq \emptyset \text{ for every } i,$$

where $r \approx 1.874$.