# On the existence of maximal group actions on compact Riemann surfaces

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# Actions of finite groups on closed oriented surfaces

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Finally, we say that an action of $Z_p$ is *finitely maximal* if and only if it does not properly extend.

# Generating vectors

A sequence $(a_1, b_1, \dots, a_h, b_h; x_1, \dots, x_r)$ of elements of a finite group $G$ is called a $(h; m_1, \dots, m_r)$-*generating vector* defining an action of $G$ on a compact surface of genus $g$ if and only if

- its elements generate $G$,
- the order of $x_i$ is equal to $m_i$,
- $x_1 \dots x_r [a_1, b_1] \dots [a_h, b_h] = 1$,
- $2(g-1) = |G| \left(2(h-1) + \sum_{i=1}^{r} (1 - 1/m_i)\right)$.

# Topologicall equivalence via generating vectors

**Theorem (J. Nielsen 1944)**

*Let*

$$(a_1, b_1, \ldots, a_h, b_h; x_1, \ldots, x_r) \quad \text{and} \quad (a'_1, b'_1, \ldots, a'_h, b'_h; x'_1, \ldots, x'_r),$$

*where $r > 0$ be two generating vectors for $Z_p$. These vectors define topologically equivalent actions of $Z_p$ if and only if there exists a permutation $\sigma \in S_r$ and an automorphism $\varphi$ of $Z_p$ such that*

$$(\varphi(x_{\sigma(1)}), \ldots, \varphi(x_{\sigma(r)})) = (x'_1, \ldots, x'_r).$$

*Moreover, there is only one, up to topological equivalence, action of $Z_p$ with the signature $(h; -)$.* □

# Necessary conditions

**Theorem (V. Peterson et al. 2017)**

*Suppose that the group $Z_p$ acts on a compact Riemann surface with the signature $(h; p, \overset{r}{\ldots}, p)$, such that $h \geq (p-3)/2$. Then, the action of $Z_p$ extends to an action of a group of order $2p$.* □

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**Theorem (V. Peterson et al. 2017)**

*Suppose that the group $Z_p$ acts on a compact Riemann surface with the signature $(h; p, \overset{r}{\ldots}, p)$, such that $h < (p-3)/2$. Moreover, suppose that the action of $Z_p$ is not finitely maximal. Then, this action extends to an action of a group $G$ of order $pq$, for some prime number $q$. Furthermore, we may assume that $Z_p$ is a normal subgroup of $G$.* □

# A criterion for being finitely maximal

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**Lemma (G. Gromadzki and J. S-T, 2026)**

*Suppose that $G$ is a group of order $pq$ containing $Z_p$ as a normal subgroup and acting on a compact Riemann surface of genus $g$. Then, the signature of the action of $G$ equals*

$$(k; q, \overset{s}{\ldots}, q, p, \overset{t}{\ldots}, p, pq, \overset{u}{\ldots}, pq),$$

*where $u \leq 2h+2$, and if $q > 2$, then $u \leq h+2$.* □

# Known sufficient condition

**Theorem (V. Peterson et al. 2017)**

*Let $p \geq 5$ be a prime, $0 \leq h \leq (p-5)/2$ and $r > p+7$ be integers. Then, there exists a finitely maximal action of $Z_p$ with the signature $(h; p, \overset{r}{\ldots}, p)$.* □

# Usefull terminology

Let

$$(1, \overset{2h}{\ldots}, 1; a_1, a_2, \ldots, a_r), \qquad (1)$$

be a generating vector for an action of $Z_p$. Recall that a permutation of $(a_1, \ldots, a_r)$ gives rise to a topologically equivalent action of $Z_p$. Thus, given a prime number $q$, we may assume that the action of $Z_p$ is defined by a vector

$$(1, \overset{2h}{\ldots}, 1; b_1, \overset{qn_1}{\ldots}, b_1, \ldots, b_t, \overset{qn_t}{\ldots}, b_t, c_1, \ldots, c_u),$$

for some $u \geq 0$, $n_1, \ldots, n_t > 0$ such that

- $b_i$ are pairwise distinct,
- each $c_i$ appears less than $q$ times,
- $u + (n_1 + \ldots + n_t)q = r$

called a *q-form* of the vector (1).

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The sequences $(b_1, \overset{qn_1}{\ldots}, b_1, \ldots, b_t, \overset{qn_t}{\ldots}, b_t)$ and $(c_1, \ldots, c_u)$ will be called the *q-block* and *q-tail* of the vector (1) respectively.

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With this terminology we have the following

**Lemma**

*Let $\tau$ be a generating vector for an action of $Z_p$, such that the length $u$ of its 2-tail does not exceed $2h + 2$. Then, the action of $Z_p$ extends to an action of $Z_{2p}$.* □

# What needs to be done

We need to determine for which triples $(p, h, r)$, there exists a sequence $(1, \overset{2h}{\ldots}, 1; a_1, a_2, \ldots, a_r)$ of elements of $Z_p$, such that

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- its $q$-tail is sufficiently long for any prime number $q$.
- the sequence $(a_1, a_2, \ldots, a_r)$ cannot be decomposed into orbits of nontrivial action of any $Z_q$ on $Z_p$,
- the sequence $(a_1, a_2, \ldots, a_r)$ cannot be permuted into a sequence of the form $(b_1, \overset{p}{\ldots}, b_1, \ldots, b_t, \overset{p}{\ldots}, b_t)$.

# Main theorem

**Theorem**

*There exists a finitely maximal action of* $Z_p$ *with the signature* $(h; p, \overset{r}{\ldots}, p)$ *if and only if one of the following conditions holds*

- $h \leq (p-9)/2$ *and* $r \geq 2h+3$,
- $h = (p-7)/2$, $p > 7$ *and* $r = 2h+3$ *or* $r \geq 2h+5$,
- $h = (p-5)/2$, $p \geq 7$ *and* $r = p, p+2$ *or* $r \geq 2h+9$,
- $h = 0$, $p = 7$ *and* $r \geq 5$,
- $h = 0$, $p = 5$ *and* $r = 7$ *or* $r \geq 9$.

□

# Sketch of the proof

**Theorem**

*Let $p \geq 7$ be a prime, $h \leq (p-7)/2$ and $r \geq 2h+5$ be integers. Then, there exists a finitely maximal action of $Z_p$ with the signature $(h; p, \overset{r}{\ldots}, p)$.*

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By the previous theorem, here we only need to consider the cases $r = 2h+3$ and $r = 2h+4$. We will choose pairwise different elements $n_1, \ldots, n_r \in \mathbb{Z}_p$ such that $(1, \overset{2h}{\ldots}, 1; a^{n_1}, a^{n_2}, \ldots, a^{n_r})$ is a well-defined generating vector. Observe that we only need to make sure that it cannot be decomposed into orbits of a nontrivial action of any $Z_q$ on $Z_p$.

# Sketch of the proof

We start by taking $n_1 = -1$, $n_2 = -2$, $n_3 = 3$. Let $S$ be a set of generators of all prime-ordered subgroups of $\mathbb{Z}_p^*$ whose orders divide $r$. We may assume that 2 and $p-3$ are not elements of $S$.

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$$n_4 = \lambda_1, \ldots, n_{3+m} = \lambda_m, n_{4+m} = -\lambda_1, \ldots, n_{3+2m} = -\lambda_m,$$

the remaining elements can be choosen from the set $B$ by taking suitably many pairs $x, -x$. $\square$

## Sketch of the proof

Consider a signature $(10; 31, \overset{24}{\ldots}, 31)$. We will construct a finitely maximal action of $Z_{31}$ with this signature. Observe that there are two prime ordered subgroups of $\mathbb{Z}_{31}^*$ whose orders divide the number of fixed points i.e. the subgroups of orders 2 and 3.

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$$n_1 = 30, n_2 = 29, n_3 = 28, n_4 = 6, n_5 = 5, n_6 = 26.$$

The remaining elements we may choose from the set $B$ by taking sufficiently many pairs $(x, -x)$.

Thank you for your attention!