Recent discoveries about finite quotients of triangle groups and quadrangle groups

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Triangle groups

For positive integers k, l and m, the ordinary (k, l, m) triangle group is the abstract group with presentation

$$\Delta^+(k, l, m) = \langle\, x, y, z \mid x^k = y^l = z^m = xyz = 1 \,\rangle$$

– or equivalently, $\langle\, x, y \mid x^k = y^l = (xy)^m = 1 \,\rangle$.

Examples:

- $\Delta^+(2, n, 2) = \langle\, x, y \mid x^2 = y^n = (xy)^2 = 1 \,\rangle \cong D_n$ (dihedral)
- $\Delta^+(2, 3, 5) \cong A_5$ (the icosahedral group)
- $\Delta^+(2, 3, 7)$ is the universal Hurwitz group.

The full (k, l, m) triangle group is the abstract group

$$\Delta(k, l, m) = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^k = (bc)^l = (ca)^m = 1 \,\rangle$$

in which $x = ab$, $y = bc$ and $z = ca$ generate a subgroup of index 2 isomorphic to $\Delta^+(k, l, m)$.

Quadrangle groups

For positive integers p, q, r and s, the ordinary (p,q,r,s) quadrangle group is the abstract group with presentation

$$\square^{+}(p,q,r,s) = \langle\, x,y,z,w \mid x^p = y^q = z^r = w^s = xyzw = 1 \,\rangle$$

– or equivalently $\langle\, x,y,z \mid x^p = y^q = z^r = (xyz)^s = 1 \,\rangle$.

Significance

Every ordinary triangle group $\Delta^{+}(k,l,m)$ is a Fuchsian group, and its 'smooth' finite quotients act on compact Riemann surfaces with signature $(0;\, k,l,m)$, and similarly, every ordinary quadrangle group $\square^{+}(p,q,r,s)$ is a Fuchsian group, and its 'smooth' quotients act on compact Riemann surfaces with signature $(0;\, p,q,r,s)$. In each case, the genus of the surface on which such a quotient acts is given by the Riemann-Hurwitz formula.

In fact, If G is a group of conformal automorphisms of a compact Riemann surface S of genus $g > 1$, and $|G| > 4(g-1)$, then G acts on S with some triangular signature (0; k, l, m) or some quadrangular signature (0; $2, 2, r, s$).

[This is easy to prove, as the Riemann-Hurwitz formula gives

$$2g - 2 = |G| \left(2\gamma - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right)$$

for actions with signature (γ; $m_1, m_2, \ldots, m_r$).]

Now by using the 'Low Index Normal Subgroups' algorithm to find all smooth quotients of ordinary triangle and quadrangle groups of the appropriate types and orders, it was possible (in 2010) to find all large conformal group actions on compact Riemann surfaces of genus 2 to 101, and the **largest** conformal group actions on compact Riemann surfaces of genus 2 to 301. [Details available on MC's website.]

Update

At a conference in June 2024, Jen Paulhus asked if I could extend the range of genera for which all 'large' conformal group actions are known. This was easy for triangular signatures $(0; 2, l, m)$, by knowledge of regular maps of small genera, and a little more difficult for triangular signatures $(0; k, l, m)$ with $3 \leq k \leq l \leq m$, but much more challenging for quadrangular signatures $(0; 2, 2, r, s)$.

The first two were handled by MAGMA's 'Low Index Normal Subgroups' procedure, but obviously needed more time and memory. The third one couldn't be done completely the same way. The cases $(0; 2, 2, 2, s)$ for odd s and the cases $(0; 2, 2, 3, s)$ for $s \in \{3, 4, 5\}$ were OK, but the cases $(0; 2, 2, 2, s)$ for even s were impossible ... causing memory overload (even on computers with over 1000Gb of space).

I thought about reducing the genus range to $1 < g \leq 201$, but even that would have been difficult, either by the usual approach, or by searching for suitable generating sets inside all groups of small order (up to 2000, with the exception of orders 1024 and 1536).

But then fortuitously I mentioned this to Primož Potočnik (visiting from U Ljubljana in Slovenia), and he said "Why not use my database of generating sets for groups of small order, consisting of three involutions, which I constructed in a search for 3-valent Cayley graphs?". This was a magic bullet — simply due to the fact that ordinary quadrangle groups $\Box^+(2,2,2s)$ are generated by three involutions.

Even then, using that information was not completely straightforward, as the database is huge, but it worked!

The outcome

We now have a complete list of all actions of 'large' groups of conformal automorphisms of a compact Riemann surface of genus g, up to natural equivalence(*), for $1 < g \leq 301$.

Moreover, in each case an indication is given as to whether or not the action is 'reflexible', in that it extends to one of a group G^* twice as large containing anti-conformal automorphisms, and in the reflexible case, whether there's a corresponding action of the given group G on a compact non-orientable quotient surface of non-orientable genus $g+1$.

For example, for quadrangular signature 0; $p.q.r.s$) the signature of the action of G^* could be $(0; +; [-]; \{(p, q, r, s)\})$, or $(0; +; [p]; (r, s))$ when $p = q$, or similar, or $(0; +; [p, r]; (1))$ when $p = q$ and $r = s$.

(*) Reflection, and/or duality in the triangular case

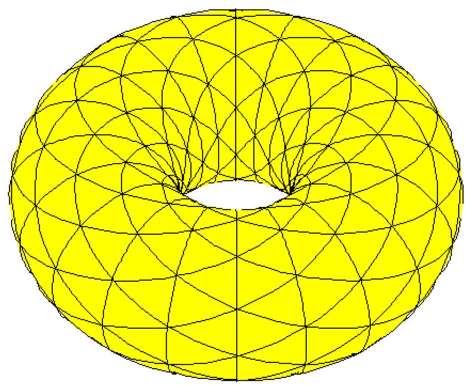
Note: Three small mistakes were encountered recently in the previous list of large actions for genus 2 to 101, as the previous tests for reflexibility and non-orientable quotients actions were not quite comprehensive enough:

- For genus 6, the largest chiral action is via a group of order 24 (acting with signature $(0; 2, 2, 3, 4)$), not order 60;
- For orientable genus 88, there are two different non-orientable quotient actions of largest possible order 696, with signatures $(0;+;[-];\{(2,2,2,4)\})$ and $(0;+;[2];\{(2,4)\})$ on a non-orientable surface of genus 89;
- For orientable genus 94, there are two different non-orientable quotient actions of largest possible order 744, with signatures $(0;+;[-];\{(2,2,2,4)\})$ and $(0;+;[2];\{(2,4)\})$ on a non-orientable surface of genus 95.

Apologies!

An unexpected outcome (for regular hypermaps)

A map is a cellular embedding of a connected graph into a surface, carving it up into simply-connected regions called its ‘faces’, and such a map is called regular if its automorphism group is as large as possible, having a single orbit on ‘flags’ ($\approx$ incident vertex-edge-face triples (v, e, f)).



A regular map of type $\{3, 6\}$ on the torus
where $3 =$ face size and $6 =$ valency

A hypermap is a generalisation of a map, which can be viewed as an embedding of a hypergraph on a surface, and a hypermap is called regular if ts automorphism group has a single orbit on its ‘brins’, which we could call ‘hyperflags’.

The automorphism group of every finite regular hypermap $\mathcal{H}$ is a smooth quotient of the full (k, l, m) triangle group for some triple $\{k, l, m\}$ (namely the ‘type’ of $\mathcal{H}$), and hence is generated by three involutions. Conversely, every smooth quotient of $\Delta(k, l, m)$ is the (full) automorphism group of some regular hypermap of type $\{k, l, m\}$.

It follows that Primož Potočnik’s database also helps extend the genus range of finite regular hypermaps.

Unfortunately this doesn’t help with chiral hypermaps. Those are harder to find than ‘large’ group actions, because the bound $|G| > 4(g-1)$ for the latter weakens to $|G| > 2(g-1)$.

Another unexpected outcome (for graph diameters)

Given integers $d \geq 3$ and $D \geq 2$, the famous degree-diameter problem involves finding the largest connected finite regular graph with degree/valency d and diameter D. The most famous examples are the Moore graphs of diameter 2, for which the natural upper bound of $1+d^2$ on the graph order is achieved only when $d = 2, 3$ or 7, or possibly 57.

This degree-diameter problem can be restricted to certain classes of graphs. For Cayley graphs, some new 'record' graphs have been found using PP's database of 3-involution generating sets, and another of his databases that was very helpful in extending the genus range of regular and chiral maps, namely one of generating pairs (x, y) for small groups in which x is an involution.

Soluble quotients of triangle groups

Over recent decades, a lot of work was devoted by many people into finding non-abelian simple quotients of ordinary triangle groups. Such quotients have included PSL$(2, q)$ [Macbeath], alternating groups [Higman/MC], and exceptional simple groups [Gareth Jones], to name just a few.

Extension of the determination of regular and chiral maps, however, threw up a big surprise: among all orientably-regular maps of genus 2 to 1501 (found by MC & PP in 2024), over 93% have a **soluble** automorphism group, and **less than 0.1%** have a non-abelian simple one. [This shows the importance that many people place on simple groups!]

My PhD student Darius Young looked into the ‘soluble’ surprise with me, and we found the following part-explanation:

Theorem [MC & Darius Young]:

Let Δ^+ be any non-perfect hyperbolic ordinary triangle group. Then

(a) Δ^+ has a smooth finite soluble quotient with derived length at most 3,

(b) if c is the minimum derived length of a smooth finite soluble quotient of Δ^+, then Δ^+ has infinitely many smooth finite soluble quotients with derived length d, for all $d > c$.

Hence in particular, Δ^+ has infinitely many smooth finite soluble quotients with derived length d, for all $d > 3$.

Our proof uses a 1970 theorem of David Singerman on the signatures of subgroups of finite index in a Fuchsian group.

Finally: **Densities of the orders of finite quotients**

About 5 years ago, Tom Tucker asked this: Given an ordinary triangle group $\Delta^+(k,l,m)$, what is the density in the positive integers of the set of orders of its finite quotients?

Tom knew this question had been considered by Coy May and Jay Zimmerman for some particular cases, and with the help of a 1976 theorem by Bertram about groups with a cyclic subgroup of large index, Tom proved that the density is 0 when one of k,l and m is coprime to the other two.

We found out later that Michael Larsen had asked the same question in a paper back in 2001, and proved (using entirely different methods) that the density is 0 for the six hyperbolic cases where $11/12 < 1/k + 1/l + 1/m < 1$, namely $(2,3,7)$, $(2,3,8)$, $(2,4,5)$, $(2,3,9)$, $2,3,10)$ and $(2,3,11)$.

In a phenomenal piece of work (involving group theory, elementary and analytic number theory, and in an early version also the probabilistic Turán-Kubilius inequality), Darius Young proved the following in his 2025 PhD thesis:

For **every** ordinary triangle group Δ^+, the density of the set of orders of finite quotients of Δ^+ is **zero**.

This theorem and its proof also had some spin-offs, obtained in recent joint work by Gabriel Verret with MC & DY:

(1) If Γ is the free product $C_k * C_l$, then the density of orders of finite quotients of Γ is 0 when at least one of k and l is odd, while the density of orders of **smooth** finite quotients of Γ is **positive** when k and l are both even.

(2) The density of orders of finite edge-transitive 3-valent graphs is 0. [This follows easily from (1), using theorems of Tutte, Goldschmidt, and Djoković & Miller.]

(3) Let Γ be **any** finitely-generated group, let $\mathcal{Q}(\Gamma)$ be the set of orders of finite quotients of Γ, and let $d(\Gamma)$ be the density of $\mathcal{Q}(F)$ in the positive integers. Then either $d(\Gamma) = 0$, or $d(\Gamma) = 1$ and Γ has an **infinite cyclic quotient**, or $d(\Gamma) = \frac{1}{2}$ and Γ has an **infinite dihedral quotient**.

Thank you for listening!

And thanks to the organisers!