

# Small Group Census of Non-maximal Actions from Singerman's List

S. Allen Broughton  
Antonio F. Costa  
Milagros Izquierdo

Rose-Hulman Institute of Technology (emeritus)  
UNED  
Linköping Universitet

AMS Eastern Special Session on Riemann Surfaces and  
Related topics

# Cast of characters

We consider a conformal actions of a finite group  $G$  on a closed hyperbolic Riemann surface  $S$  of genus  $\sigma$ .

Here are the main players throughout the talk:

- $S$  is a hyperbolic Riemann surface, i.e.,  $\sigma \geq 2$ , with standard hyperbolic metric of constant curvature  $-1$ .
- $\text{Aut}(S)$  is the group of conformal automorphisms of  $S$ . This finite group preserves the complex curve structure of  $S$  as well as the metric.
- $G < H$  is a pair of finite groups with associated Fuchsian group pair  $\Gamma < \Delta$ .
- $G$  is known and acts *conformally* on  $S$ , with signature  $\varepsilon = \varepsilon(G)$ , via a monomorphism

$$\epsilon : G \rightarrow \text{Aut}(S). \tag{1}$$

# Problem statement

Given the action  $\epsilon : G \rightarrow \text{Aut}(S)$ , a question with a long history and, as of yet, no complete answer is:

- *When is  $\epsilon(G) < \text{Aut}(S)$ ?*
- If  $\epsilon(G) < \text{Aut}(S)$  holds, the action of  $G$  is called (*finitely non-maximal*), otherwise it is (*finitely maximal*).
- Alternatively, is there a finite index overgroup  $H > G$ , such that  $\epsilon$  extends to a conformal action  $\epsilon^e : H \rightarrow \text{Aut}(S)$ .
- If the answer is yes, we write  $(G, \epsilon) < (H, \epsilon^e)$ , and say that the  $G$  action is *extendible*.

**A census of extendible actions.** We consider potential action extension pairs  $(G, \epsilon) < (H, \epsilon^e)$  corresponding to *Singerman's list*. Using Magma, we propose to perform a census of many small groups  $G$ , determining possible actions, and whether they are maximal or not.

# Motivation and Equisymmetric Stratification

**Motivation:** Though there are many applications of the solution of our problem, our primary motivation is to analyze the orbifold structure of the moduli space of surfaces of a fixed genus.

**Equisymmetric Stratification:**  $\mathcal{M}_\sigma$ , the moduli space of genus  $\sigma$  is the complex orbifold of conformal equivalence classes of Riemann surfaces of genus  $\sigma$ . The moduli space is a disjoint union of smooth, irreducible, quasi-projective *strata*. Two surfaces are in the same stratum if their automorphism groups are *topologically equivalent*. Hence, the decomposition is called the *equisymmetric stratification*.

Each stratum may be identified with a finitely maximal group action in the given genus – with suitable equivalence relations. Therefore, distinguishing between the maximal and non-maximal actions is a first step in enumeration of the strata.

# Previous work

See References (2) for more bibliographic details.

**Census for one signature** The authors completed a census for several thousand small groups, determining actions  $G$  on surfaces  $S$  such that  $S/G$  is a torus with one cone point. If the action extends, then  $G \triangleleft H$  of index 2. The proportions of maximal and non-maximal actions were both substantial.

**Extension Criteria** Research on extension of actions for Singerman's list have been conducted by these authors.

- Emilio Bujalance, Marston Conder, and Javier Cirre
- Antonio Costa and Hugo Parlier

**Equisymmetric Stratification and Branch locus** Originally introduced by Bill Harvey and described in detail by SAB.

# The quotient $S/G$

- The orbit space  $S/G = S/\epsilon(G)$  is a topological surface  $T = S/G$  of *orbit genus*  $\tau$ .
- The hyperbolic metric on  $S$  descends to a hyperbolic metric on  $T$  except at the cone points  $w_1, \dots, w_t$  of orders  $n_1, \dots, n_t$ , which must be orders of non-trivial elements of  $G$ .
- In local coordinates  $\pi_G : S \rightarrow T$  has the form  $w = z^{n_i}$  at the points lying over  $w_i$
- $\mathfrak{s} = \mathfrak{s}(G, \epsilon) = (\tau; n_1, \dots, n_t)$  is called the *signature* of the  $G$  action. We also call it the signature,  $\mathfrak{s}(T)$ , of  $T$ .
- The Riemann-Hurwitz equation relates the group order  $|G|$ , the genus of  $S$  and the signature:

$$\frac{2\sigma - 2}{|G|} = 2\tau - 2 + \sum_{i=1}^t \left(1 - \frac{1}{n_i}\right). \quad (2)$$

# Moduli space of hyperbolic orbifolds

As noted, our quotient orbifolds  $T$ , are Riemann surfaces of genus  $\tau$ , with  $t$  cone points, with prescribed orders, and so the signature is:

$$\mathfrak{s} = \mathfrak{s}(T) = (\tau; n_1, \dots, n_t). \quad (3)$$

- There is a moduli space  $\mathcal{M}_{\mathfrak{s}}$ , consisting of conformal equivalence classes of orbifolds of genus  $\tau$  with  $t$  cone points, and the given orders.
- Lying over  $\mathcal{M}_{\mathfrak{s}}$ , is a Teichmüller space  $\mathcal{T}_{\mathfrak{s}} \rightarrow \mathcal{M}_{\mathfrak{s}}$ , a contractible branched covering space.
- The common dimension of these spaces is given by

$$d_{\mathfrak{s}} = 3\tau - 3 + t \quad (4)$$

# Relative projections

- Assume we have a finite group pair  $G < H$  and actions

$$\epsilon : G \rightarrow \text{Aut}(S), \quad \epsilon^e : H \rightarrow \text{Aut}(S)$$

such that  $\epsilon^e$  extends  $\epsilon$  from  $G$  to  $H$ .

- Set  $T = S/G$ ,  $U = S/H$  and define the *relative projection*

$$\pi_{G/H} : T \rightarrow U, \tag{5}$$

to be the branched cover of orbifolds, of degree  $n = |H/G|$ , induced by  $Gx \rightarrow Hx$ .

- Assume the signatures:  $\mathfrak{s}(T) = \mathfrak{s}(G) = (\tau; n_1, \dots, n_t)$  and  $\mathfrak{s}(U) = \mathfrak{s}(H) = (\rho; m_1, \dots, m_r)$ . In the next two slides we show, by example, how we may summarize how the relative projection  $\pi_{G/H}$  numerically intertwines the two signatures.

# Klein's quartic - 1

## Example (Klein's quartic)

Consider the well known example of a pair of index 24

$$G = \mathbb{Z}_7 < PSL(2, 7) = H,$$

acting upon  $S =$  Klein's quartic of genus 3. We have

- $\mathfrak{s}(G) = (0; 7, 7, 7)$  and  $\mathfrak{s}(H) = (0; 2, 3, 7)$
- Let  $w_1, w_2, w_3$  be the cone points of orders 2, 3, and 7, respectively, on  $U = S/H$ .
- The fibre  $\pi_{H/G}^{-1}(w_1)$  consists of 12 ordinary points on  $T$ . The ramification of  $\pi_{H/G}$  locally looks like  $w = z^2$ .
- represent the fibre over  $w_1$  by  $1^{12}$  and the ramification of  $\pi_{H/G}$  on this fibre by  $2^{12}$ .

# Klein's quartic - 2

## Example (Klein's quartic continued)

- For  $w_2$  we get  $1^8$  for the fibre and  $3^8$  for the ramification.
- Over  $w_3$  there are three cone points of order 7 and three ordinary points. The fibre is denoted  $7^3 \cdot 1^3$  and the ramification by  $1^3 \cdot 7^3$ .

The forgoing is summarized by a *numerical projection*

$$\nu : \mathfrak{s}(G) \rightarrow \mathfrak{s}(H), \quad (6)$$

specifically:

$$(2^{12}, 3^8, 1^3 \cdot 7^3) : (0; 1^{12}, 1^8, 7^3 \cdot 1^3) \rightarrow (0; 2, 3, 7) \quad (7)$$

## Uniformization of actions - SKE's and GV's

- For a hyperbolic orbifold  $T$  with signature  $\mathfrak{s}$  there is a Fuchsian group  $\Gamma = \Gamma_{\mathfrak{s}} = \Gamma_T \simeq \pi_1^{orb}(T)$ , with the same signature and  $T \simeq \mathbb{H}/\Gamma$ , as hyperbolic orbifolds.
- $\Gamma$  has a *canonical generating set*  $\mathcal{X}$  yeilding a presentation:

$$\Gamma = \langle \alpha_1, \dots, \alpha_{\tau}, \beta_1, \dots, \beta_{\tau}, \gamma_1, \dots, \gamma_t : \quad (8)$$

$$\prod_{i=1}^{\tau} [\alpha_i, \beta_i] \prod_{j=1}^t \gamma_j = \gamma_1^{n_1} = \dots = \gamma_t^{n_t} = 1 \rangle. \quad (9)$$

- Given a torsion free Fuchsian group  $\Pi \simeq \pi_1(S)$ , the *surface kernel epimorphism (SKE)*  $\Pi \hookrightarrow \Gamma \xrightarrow{\eta} G$  defines the *generating vector*  $\mathcal{V} = \eta(\mathcal{X})$  of  $G$  by:

$$\mathcal{V} = (a_1, \dots, a_{\tau}, b_1, \dots, b_{\tau}, c_1, \dots, c_t) \quad (10)$$

$$a_i = \eta(\alpha_i), b_i = \eta(\beta_i), c_j = \eta(\gamma_j).$$

# Uniformization of actions - generating vectors

- Any “vector”  $\mathcal{V} = (a_1, \dots, a_\tau, b_1, \dots, b_\tau, c_1, \dots, c_t)$  of elements of  $G$ , generating  $G$ , satisfying the relations implied by (9) is called a  $\mathfrak{s}$ -*generating vector*.
- Upon fixing  $\mathcal{X}$ , there is a 1-1 correspondence between  $\mathfrak{s}$ -generating vectors of  $G$  and SKE's  $\eta : \Gamma_{\mathfrak{s}} \rightarrow G$ .

## Proposition (Realizability of actions)

*Given a hyperbolic orbifold  $T$  and generating vector  $\mathcal{V}$  of the same signature  $\mathfrak{s}$ , then there is an SKE  $\eta : \Gamma_T \rightarrow G$  with kernel  $\Pi$  such that:*

- *$G$  acts conformally upon  $S = \mathbb{H}/\Pi$ , via  $\bar{\eta}^{-1} : G \rightarrow \Gamma_T/\Pi$ , with signature  $\mathfrak{s}$ , and*
- *$S/G \simeq T$  as hyperbolic orbifolds.*

## Equivalence of actions

- Assume a given SKE  $\eta : \Gamma_{\mathfrak{s}} \rightarrow G$ , with kernel  $\Pi$ , and the dualized  $(G, \mathfrak{s})$ -action  $\epsilon : G \rightarrow \text{Aut}(S)$ , where  $S = \mathbb{H}/\Pi$ ,  $T = \mathbb{H}/\Gamma_{\mathfrak{s}} = S/G$ , and  $\epsilon = \bar{\eta}^{-1}$ .
- The group  $\text{Aut}(G) \times \text{Aut}(\Gamma_{\mathfrak{s}})$  acts upon the SKE's:

$$\eta \rightarrow \omega \circ \eta \circ \phi^{-1}, \text{ for } (\omega, \phi) \in \text{Aut}(G) \times \text{Aut}(\Gamma_{\mathfrak{s}}) \quad (11)$$

### Proposition (Topological equivalence)

- *The action of  $\text{Aut}(G) \times \text{Aut}(\Gamma_{\mathfrak{s}})$  transfers to the generating vectors, and to the  $(G, \mathfrak{s})$ -actions on  $S$  with  $S/G = T$*
- *The SKE's in the  $\text{Aut}(G)$  orbit of  $\eta$  all define the same surface  $S$  and quotient map  $\pi_G : S \rightarrow T$ .*
- *The  $\text{Aut}(\Gamma)$  orbit of  $\eta$  permutes the (finitely many) surfaces lying over  $T$ , i.e., those surfaces  $S$  with a  $(G, \mathfrak{s})$ -action and  $S/G \simeq T$ . They are called **modular companions** of  $S$ .*

# Uniformization of action extension pairs

Given the extension  $(G, \epsilon) < (H, \epsilon^e)$  there is a triple of Fuchsian groups,  $\Pi < \Gamma < \Delta$  satisfying the following:

- $\Pi \triangleleft \Gamma$  and  $\Pi \triangleleft \Delta$ .
- $\Pi$  is torsion free and  $S \simeq \mathbb{H}/\Pi$
- $\Gamma/\Pi$  and  $\Delta/\Pi$  act naturally and conformally on  $S \simeq \mathbb{H}/\Pi$ , and may be considered as subgroups of  $\text{Aut}(S)$ .
- There is an SKE  $\Pi \hookrightarrow \Gamma \xrightarrow{\eta} G$  and an extension  $\eta^e : \Delta \rightarrow H$ .
- $\bar{\eta} : \Gamma/\Pi \rightarrow G$  is the inverse of  $\epsilon$  and the similarly for the extensions.
- We define  $s(\Gamma) = s(T) = s(G)$  and  $s(\Delta) = s(U) = s(H)$

# Monodromy and realizability - 1

Suppose we have action pair of type  $(G, \mathfrak{s})$  and  $(H, \mathfrak{s}')$  and that  $q : T \rightarrow U$  is a branched covering of the corresponding orbifolds  $T = S/G$  and  $U = S'/H$  such that:

- The map  $q$  respects the cone-point structure on  $T$  and  $U$  as in the Klein quartic example (6) and (7).
- $w_1, \dots, w_r$  are the cone points on  $U$  and  $w_0$  is a regular point on  $U$

Then:

- **Local monodromy:** Traveling counter-clockwise along a small circle about the cone point  $w_j$  induces a permutation  $p_j$  of the points in  $q^{-1}(w_0)$ , lying over the regular value  $w_0$ .
- **Path dependence:** The permutation  $p_j$  is determined up to conjugacy by a path from  $w_0$  to  $w_j$  and the labelling of  $q^{-1}(w_0)$ . The partition structure is given by the ramification degrees of  $q$ . See (7) for example.

## Monodromy and realizability - 2

### Theorem (Riemann's Existence Theorem)

Assume that the **hyperbolic orbifold**  $U$  has **genus 0**. Then a numerical projection  $\nu$  as in (6) is realizable as a branched cover  $\pi_{H/G} : T \rightarrow U$  if and only if there is a sequence of permutations  $p_1, \dots, p_r$  such that

- the partition structure of  $p_j$  is determined by the ramification over  $w_j$ .
- the product  $p_1 \cdots p_r = 1$ , and
- the monodromy group  $M_\nu = \langle p_1, \dots, p_r \rangle$  is a transitive subgroup of  $\Sigma_n$ .

The hyperbolic orbifold  $T$  lying over  $U$  is determined, up to conformal equivalence, by the hyperbolic structure of  $U$  and the monodromy vector  $(p_1, \dots, p_r)$ .

# Codimension and Singerman's List

- Given a numerical projection (6) the codimension of the projection is

$$d_{\mathfrak{s}(G)} - d_{\mathfrak{s}(H)} = d_{\mathfrak{s}(\Gamma)} - d_{\mathfrak{s}(\Delta)} = 3(\tau - \rho) + t - r. \quad (12)$$

- **Singerman's List is the set of all realizable numerical projections of codimension 0.**
- The Klein's quartic example (7) is on Singerman's list and is a *constrained projection*.
- For  $u \geq 4$ ,  $(3, 3, 1^3) : (0; 1, 1, u^3) \rightarrow (0 : 3, 3, u)$  gives a *parametric family* of projections.
- Singerman's list consists of a finite number of *constrained projections* and a finite number *parametric families* of projections.

# Normality of extensions, monodromy group

Given a uniformizing pair,  $\Gamma < \Delta$  for a plausible extension pair  $(G, \epsilon) < (H, \epsilon^e)$ , the extension criteria depend on these two cases: 1) normal inclusion:  $\Gamma \triangleleft \Delta$  and 2) non-normal inclusion:  $\Gamma \not\triangleleft \Delta$

Furthermore:

- The extension criteria are known for Singerman's list.
- Assuming the extension exists, the coset spaces  $\Delta/\Gamma$  and  $H/G$  determine conjugate subgroups of  $\Sigma_n$ .
- If  $\rho = 0$ , both of the subgroups above are conjugate to the monodromy group  $\text{Mon}(\Delta/\Gamma) = M_\nu = \langle p_1, \dots, p_r \rangle$ .

The methods to find extension criteria work in higher codimension.

## Some lines from Singerman's List

We do not list all the 19 cases in Singerman's list. Here is a portion of the list, illustrating the intricacies of the extension problem. (Case numbering taken from Bujalance et al.)

|     | $s(\Gamma)$               | $s(\Delta)$        | $ \Delta/\Gamma $ | $\Gamma \triangleleft \Delta$ | $\text{Mon}(\Delta/\Gamma)$ |
|-----|---------------------------|--------------------|-------------------|-------------------------------|-----------------------------|
| N3  | $(1; u), u \geq 2$        | $(0; 2, 2, 2, 2u)$ | 2                 | yes                           | $\mathbb{Z}_2$              |
| N6  | $(0; u, u, u), u \geq 4$  | $(0; 3, 3, u)$     | 3                 | yes                           | $\mathbb{Z}_3$              |
| N7  | $(0; u, u, u), u \geq 4$  | $(0; 2, 3, 2u)$    | 6                 | yes                           | $\Sigma_3$                  |
| T1  | $(0; 7, 7, 7)$            | $(0; 2, 3, 7)$     | 24                | no                            | $PSL(2, 7)$                 |
| T2  | $(0; 2, 7, 7)$            | $(0; 2, 3, 7)$     | 9                 | no                            | $PSL(2, 8)$                 |
| T3  | $(0; 3, 3, 7)$            | $(0; 2, 3, 7)$     | 8                 | no                            | $PSL(2, 7)$                 |
| T5  | $(0; 3, 8, 8)$            | $(0; 2, 3, 8)$     | 10                | no                            | $PSL(2, 9)$                 |
| T7  | $(0; 4, 4, 5)$            | $(0; 2, 4, 5)$     | 6                 | no                            | $PSL(2, 5)$                 |
| T10 | $(0; 3, u, 3u), u \geq 2$ | $(0; 2, 3, 3u)$    | 6                 | no                            | $Alt_4$                     |

## Census considerations for "The List"

Here are three considerations for extension problems coming from Singerman's list:

- normal versus non-normal inclusions  $\Gamma < \Delta$ , identified in the Table, different extension methods
- constrained inclusions versus parametric families of inclusions, in parametric case parameters are filled in from orders in  $G$
- non-primitive inclusions i.e., there are intermediate groups  $\Gamma < \Gamma_1 < \Delta$ . Lines N7 and T1 are imprimitive.

# Census Process Outline - 1

- Each constrained inclusion,  $\Gamma < \Delta$ , or family of inclusions, is treated as a separate, systematic collection of calculations.
- For the purpose of proving non-maximality we need only consider primitive extensions.
- Once an inclusion  $\Gamma < \Delta$  is selected, we work through a selection of small groups, in their order in the Small Groups database in MAGMA.
- Some groups  $G$  may be skipped over on the basis of their order or structural reasons.

## Census Process Outline - 2

Select a pair  $(G, \mathfrak{s})$  and compute all action data for the pair.

- Find the set of  $(G, \mathfrak{s})$  generating vectors by brute force.
- From each  $\text{Aut}(G)$  orbit of generating vectors, a unique normal form is selected, reducing computational complexity,
- Using the  $\text{Aut}(\Gamma)$  action the normal forms are grouped into *modular companion orbits*. A normal form for each orbit and the size of each orbit is computed.
- The group identification, the signature, the genus of  $S$ , the  $\text{Aut}(G) \times \text{Aut}(\Gamma_{\mathfrak{s}})$  representatives, and the modular companion orbit sizes are recorded.

## Census Process Outline - 3

For candidate overgroups, consider all small groups  $H$  of order  $n \times |G|$ .

- Check that there is an inclusion  $G < H$ , if not, eliminate  $H$ .
- The permutation representation  $q_H : H \rightarrow \text{Mon}(\Delta/\Gamma)$  is surjective. Eliminate groups  $H$  without this structure.
- For each candidate  $H$  compute all  $(H, \mathfrak{s}^e)$  action data,
- For each  $\text{Aut}(H)$  class of actions check if the associated SKE  $\eta'$  restricts to an SKE  $\eta$  of  $G$ . Different methods for the normal and non-normal case.
- Record the  $\eta$  so found, these are non-maximal actions.

# Sample Results

From the authors' previous work:

| Case ID | max $G$ | # groups | # actions | % Max | % Non-max |
|---------|---------|----------|-----------|-------|-----------|
| N3      | 200     | 6064     | 26444     | 21.9  | 78.1      |

Prime order group  $G = \mathbb{Z}_p$ ,  $p \geq 5$  and signature  $(0; p, p, p)$

- computable by hand
- $p - 2$  actions on surfaces of genus  $\frac{p-1}{2}$
- 2 extensions to  $H = \mathbb{Z}_3 \times \mathbb{Z}_p$  if 3 divides  $p - 1$
- 3 extensions to  $H = \mathbb{Z}_2 \times \mathbb{Z}_p$
- remaining actions not extensible

# Thanks for listening

# Any Questions

# References

- S.A. Broughton, A.F. Costa, M. Izquierdo, *One Dimensional Equisymmetric Strata in Moduli Space with Genus 1 Quotient Surfaces*, Rev. Real Acad. Cienc. <https://doi.org/10.1007/s13398-023-01520-9>
- S.A. Broughton, A.F. Costa, M. Izquierdo *Modular Companions*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2024) 118:21
- S.A. Broughton, *The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups*, Topology Appl. 37 (1990) 101-113.
- E. Bujalance, F.J. Cirre, M. Conder, *On extendability of group actions on compact Riemann surfaces*, Trans. Amer. Math. Soc. 355 (2002), 1537-1557.
- A.F. Costa and H. Parlier, *Applications of a theorem of Singerman about Fuchsian groups*, (2008)