HOLOMORPHIC FAMILIES OF JORDAN CURVES

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Notations. We will use the following notations:

$\mathbb{C}$ for the complex plane, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for the Riemann sphere, and Δ for the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$.

Definition

A complex-valued function $w = f(z)$ defined on a region Ω in $\mathbb{C}$ is called a quasiconformal mapping if it is a sense–preserving homeomorphism of Ω onto its image and its complex distributional derivatives

$$w_z = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big) \text{ and } w_{\overline{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)$$

are Lebesgue measurable locally square integrable functions on Ω that satisfy the inequality $|w_{\overline{z}}| \leq k|w_z|$ almost everywhere in Ω, for some real number k with $0 \leq k < 1$.

If $w = f(z)$ is a quasiconformal mapping defined on the region Ω then the function w_z is known to be nonzero almost everywhere on Ω. Therefore the function

$$\mu_f = \frac{w_{\overline{z}}}{w_z}$$

is a well–defined L^∞ function on Ω, called the *complex dilatation* or the *Beltrami coefficient* of f. The L^∞ norm of every Beltrami coefficient is less than one.

The positive number

$$D_f = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}$$

is called the *dilatation* of f. We say that f is K-quasiconformal if f is a quasiconformal mapping and $D_f \leq K$.

We call a homeomorphism of $\widehat{\mathbb{C}}$ *normalized* if it fixes the points 0, 1, and ∞.

Theorem (Existence Theorem)

Let $M(\mathbb{C})$ denote the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Then, for each μ in $M(\mathbb{C})$, there exists a unique normalized quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ onto itself that has Beltrami coefficient μ; we denote this quasiconformal map by w^μ.

The basepoint of $M(\mathbb{C})$ is the zero function.

Furthermore, we have the following important theorem:

Theorem (Ahlfors and Bers, 1960)

For every fixed $z \in \mathbb{C}$, the map $\mu \mapsto w^\mu(z)$ of $M(\mathbb{C})$ into $\mathbb{C}$ is holomorphic.

In their study of the dynamics of rational maps Mañe, Sad, and Sullivan introduced the concept of holomorphic motions.

Definition (Mañe, Sad, and Sullivan, 1983)

Let $E \subset \widehat{\mathbb{C}}$. A map $\phi : \Delta \times E \to \widehat{\mathbb{C}}$ is called a holomorphic motion if it satisfies the following properties:
(i) $\phi(0, z) = z$ for all $z \in E$,
(ii) For each $t \in \Delta$, $\phi(t, \cdot) : E \to \widehat{\mathbb{C}}$ is injective as a function from E to $\widehat{\mathbb{C}}$ (ie ϕ is injective on E), and
(iii) For each $z \in E$, $\phi(\cdot, z) : \Delta \to \widehat{\mathbb{C}}$ is holomorphic (ie ϕ is holomorphic in Δ).

We sometimes write $\phi_t(z)$ for $\phi(t, z)$. As t moves in Δ, the set $E_t := \phi_t(E)$ moves in $\widehat{\mathbb{C}}$. We can think of t as the complex time-parameter for the motion ϕ, and call Δ the parameter space of ϕ. We say that ϕ is a holomorphic motion of E over Δ.

Henceforth, we will always write λ for t (in Δ).

The classical λ-lemma of Mañé, Sad, and Sullivan states three surprising properties of holomorphic motions.

The λ-lemma. (Mañé, Sad, and Sullivan, 1983) Let $\phi : \Delta \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. Then

- (i) ϕ is a continuous map from $\Delta \times E$ to $\widehat{\mathbb{C}}$,
- (ii) the map $z \mapsto \phi(\lambda, z)$ is the restriction to E of a quasiconformal mapping of the sphere.
- (iii) ϕ extends to a holomorphic motion $\widehat{\phi}$ of the closure $\overline{E}$ of E.

Remark. In the study of holomorphic motions, it is usually assumed that 0, 1, and ∞ belong to E and that the motion ϕ is normalized; i.e. 0, 1, and ∞ are fixed points of the map $\phi(\lambda, \cdot)$ for every λ in Δ.

This can always be achieved after a suitable change of coordinates: choose three distinct points a, b, c in E. For each $\lambda \in \Delta$, let S_λ be the Möbius transformation that maps $\phi_\lambda(a)$, $\phi_\lambda(b)$, $\phi_\lambda(c)$ to 0, 1, ∞ respectively. Let $\widehat{E} = S_0(E)$. Then the map $\widehat{\phi} : \Delta \times \widehat{E} \to \widehat{\mathbb{C}}$ defined by

$$\widehat{\phi}(\lambda, S_0(z)) = S_\lambda(\phi(\lambda, z))$$

is a normalized motion.

An important topic in the study of holomorphic motions is the question on extensions of holomorphic motions.

Let $E \subset \widehat{E} (\subset \widehat{\mathbb{C}})$. If $\phi : \Delta \times E \to \widehat{\mathbb{C}}$ and $\widehat{\phi} : \Delta \times \widehat{E} \to \widehat{\mathbb{C}}$ are two holomorphic motions, we say that $\widehat{\phi}$ extends ϕ if

$$\widehat{\phi}(\lambda, z) = \phi(\lambda, z) \qquad \text{for all } (\lambda, z) \in \Delta \times E.$$

A fundamental result in the study of holomorphic motions is the following theorem:

Theorem (**Slodkowski, 1991**)

Let $\phi : \Delta \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. Then, there exists a holomorphic motion $\widehat{\phi} : \Delta \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, such that $\widehat{\phi}$ extends ϕ.

Here is a more general definition of holomorphic motions.

Definition

Let V be a connected complex manifold with a basepoint x_0 and let E be a subset of the Riemann sphere $\widehat{\mathbb{C}}$. A **holomorphic motion of E over V** is a map $\phi\colon V \times E \to \widehat{\mathbb{C}}$ that has the following three properties:

(a) $\phi(x_0, z) = z$ for all z in E,

(b) the map $\phi(x, \cdot)\colon E \to \widehat{\mathbb{C}}$ is injective for each x in V, and

(c) the map $\phi(\cdot, z)\colon V \to \widehat{\mathbb{C}}$ is holomorphic for each z in E.

We will write $\phi(x, z)$ as $\phi_x(z)$ for x in V and z in E.

We say that V is the **parameter space** of the holomorphic motion ϕ.
We will always assume that ϕ is a **normalized** holomorphic motion; i.e. 0, 1, and ∞ belong to E and are fixed points of the map $\phi_x(\cdot)$ for every x in V.

Quasicircle. A **quasicircle** is a quasiconformal image of a circle.

Jordan curve. A **closed Jordan curve** is a homeomorphism of a circle. A domain whose boundary is a Jordan curve is called a Jordan domain.

Let γ_0 be a closed Jordan curve in $\widehat{\mathbb{C}}$ and let $\phi : \Delta \times \gamma_0 \to \widehat{\mathbb{C}}$ be a holomorphic motion; here, the basepoint of the parameter space Δ is obviously 0. For each $\lambda \in \Delta$, by the λ-lemma,

$$\gamma_\lambda = \{\phi_\lambda(z) : z \in \gamma_0\}$$

is a closed Jordan curve in $\widehat{\mathbb{C}}$.

Consider the family $\{\gamma_\lambda : \lambda \in \Delta\}$ generated by a holomorphic motion $\phi : \Delta \times \gamma_0 \to \widehat{\mathbb{C}}$.

Theorem (Gehring and Pommerenke,1984)

If γ_0 is a quasicircle, then each γ_λ is also a quasicircle.

Later, Pommerenke and Rodin proved the following generalization.

Theorem (Pommerenke and Rodin,1986)

Let $\{\gamma_\lambda : \lambda \in \Delta\}$ be generated by a holomorphic motion of γ_0. For each $\lambda \in \Delta$ there exists a quasiconformal homeomorphism $\widetilde{\phi}_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which maps γ_0 onto γ_λ.

The following is a consequence of Slodkowski's extension theorem.

The Beltrami coefficient μ_λ of $\widetilde{\phi}_\lambda$ varies holomorphically with respect to λ in Δ, and $\|\mu_\lambda\|_\infty \le |\lambda|$.

Here is a much stronger version of the theorem of Pommerenke and Rodin.

Theorem A. [Xinlong Dong, Arshiya Farhath. G and Mitra, 2026] Let V be a simply connected complex Banach manifold with a basepoint x_0. Let γ_{x_0} be a closed Jordan curve, and let $\phi : V \times \gamma_{x_0} \to \widehat{\mathbb{C}}$ be a holomorphic motion. We may assume that 0, 1, and ∞ belong to γ_{x_0}. Let $\{\gamma_x : x \in V\}$ be generated by the holomorphic motion ϕ. Then we have the following:

(i) For each x in V, there is a quasiconformal map $\widetilde{\phi}_x : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which maps γ_{x_0} onto γ_x.

(ii) The Beltrami coefficient μ_x of $\widetilde{\phi}_x$ varies continuously with respect to x in V.

(iii) The L^∞ norm of μ_x is bounded above by a number less than 1, that depends only on the Kobayashi distance from x to x_0, denoted by $\rho_V(x, x_0)$.

Recall that for any complex manifold X, the Kobayashi pseudometric ρ_X is the largest pseudometric such that $\rho_X(f(z_1), f(z_2)) \leq \rho_\Delta(z_1, z_2)$ where ρ_Δ is the Poincaré (hyperbolic) metric on Δ, and $f : \Delta \to X$ is any holomorphic map. The Poincaré metric on Δ is given by:

$$\rho_\Delta(z_1, z_2) = \tanh^{-1}\left|\frac{z_1 - z_2}{1 - z_1\bar{z_2}}\right|.$$

In the above theorem we assume that V is Kobayashi-hyperbolic, which means, that ρ_V is a metric.

Conjecture. The Beltrami coefficient μ_x of $\widetilde{\phi}_x$ varies real-analytically with respect to x in V.

In an ongoing joint work with Professor Yunping Jiang, we are exploring this conjecture.

We have obtained a weaker version of this conjecture.

Theorem B. [Xinlong Dong, Arshiya Farhath. G and Mitra, 2026]
Let V be a simply connected complex Banach manifold with a basepoint x_0. Let γ_{x_0} be a closed Jordan curve, and let E be a finite subset of γ_{x_0}, containing the points 0, 1, and ∞. Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion and let $E_x := \phi_x(E)$. Then we have the following:

(i) For each x in V, there is a quasiconformal map $\widetilde{\phi}_x : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widetilde{\phi}_x(E) = E_x$, and $\widetilde{\phi}_x(\gamma_{x_0})$ is a closed Jordan curve γ_x, and $E_x \subset \gamma_x$.

(ii) The Beltrami coefficient μ_x of $\widetilde{\phi}_x$ varies real-analytically on V.

(iii) The L^∞ norm of μ_x is bounded above by a number less than 1, that depends only on the Kobayashi distance from x to x_0, denoted by $\rho_V(x, x_0)$.

Our results depend on the properties of some generalized Teichmüller spaces. Let E be a closed set in the Riemann sphere $\widehat{\mathbb{C}}$, such that 0, 1, and ∞ belong to E.

Recall that a homeomorphism of $\widehat{\mathbb{C}}$ is called *normalized* if it fixes the points 0, 1, and ∞.

Definition

Two normalized quasiconformal self-mappings f and g of the Riemann sphere $\widehat{\mathbb{C}}$ are said to be E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The *Teichmüller space* $T(E)$ is the set of all E-equivalence classes of normalized quasiconformal self-mappings of $\widehat{\mathbb{C}}$.

By "$f^{-1} \circ g$ is isotopic to the identity rel E" we mean that, at each stage of the isotopy, $f^{-1} \circ g$ keeps the set E pointwise fixed.

The basepoint of $T(E)$ is the E-equivalence class of the identity map.

Let $M(\mathbb{C})$ be the open unit ball of the complex Banach space $L^{\infty}(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal map w^{μ} of $\widehat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

We define the quotient map $P_E : M(\mathbb{C}) \to T(E)$ by setting $P_E(\mu)$ equal to the E-equivalence class of w^{μ}, written as $[w^{\mu}]_E$. Clearly, P_E maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$.

In his 1990 Cornell University doctoral dissertation, G. S. Lieb proved that $T(E)$ is a complex Banach manifold such that the projection map $P_E : M(\mathbb{C}) \to T(E)$ is a holomorphic split submersion; this means, the map P_E is surjective, holomorphic, and has local holomorphic sections.

Our theorems crucially depend on the following:

Theorem. There is a continuous basepoint preserving map s from $T(E)$ to $M(\mathbb{C})$ such that $P_E \circ s$ is the identity map on $T(E)$.

Definition

The map s from $T(E)$ to $M(\mathbb{C})$ is called the *Douady-Earle section* of P_E for the Teichmüller space $T(E)$.

Remark. If E is a finite set, it can be shown that the map s is real-analytic. This enables us to prove Theorem B. The conjecture that we mentioned after Theorem A depends on extending this fact to infinite sets.

THANK YOU VERY MUCH!

Appendix 1.

Let $w = f(z)$ $(z = x + yi, w = u + vi)$ be a C^1 homeomorphism from one region to another. At a point z_0 it induces a linear mapping of the differentials

$$du = u_x dx + u_y dy \qquad \text{and} \qquad dv = v_x dx + v_y dy$$

which we can write in the complex form

$$dw = f_z dz + f_{\overline{z}} d\overline{z}$$

with

$$f_z = \frac{1}{2}(f_x - if_y) \qquad \text{and} \qquad f_{\overline{z}} = \frac{1}{2}(f_x + if_y).$$

Geometrically, it maps circles about the origin to ellipses (from the (dx, dy) plane to the $((du, dv)$ plane). We have

$$|f_z|^2 - |f_{\overline{z}}|^2 = u_x v_y - u_y v_x = J$$

which is the Jacobian. For a sense-preserving map $J > 0$ and so $|f_{\overline{z}}| < |f_z|$.

We have

$$(|f_z| - |f_{\overline{z}}|)|dz| \leq |dw| \leq (|f_z| + |f_{\overline{z}}|)|dz|.$$

The ratio of the major axis to the minor axis is

$$D_f = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \geq 1$$

called the dilatation at the point z.

The complex dilatation is

$$\mu_f = \frac{f_{\overline{z}}}{f_z}.$$

Note that we have $|\mu_f| < 1$.

Definition. The mapping f is called **quasiconformal** if D_f is bounded. It is called K-quasiconformal if $D_f \leq K$.

Remark. If $\mu_f = 0$, and equivalently, $D_f = 1$, the map f is conformal.

Appendix 2. A continuous map $f : D \to \mathbb{C}$, where D is a region in $\mathbb{C}$, has L^p "distributional" (or "generalized") derivatives ($p \geq 1$) on D if and only if there exist measurable functions f_z and $f_{\overline{z}}$, locally (ie on compact sets) L^p integrable in D, such that for each "test function" $\varphi \in C_0^1(D)$ (C^1 functions with compact support on D), the following relations hold:

$$\int_D (f\varphi_z + f_z\varphi)dxdy = 0$$

and

$$\int_D (f\varphi_{\overline{z}} + f_{\overline{z}}\varphi)dxdy = 0.$$

If the function f has locally L^p generalized derivatives as defined above, then actually f must possess usual derivatives f_x and f_y a.e. on D; moreover $\frac{1}{2}(f_x + if_y)$ and $\frac{1}{2}(f_x - if_y)$ will respectively equal the generalized derivatives f_z and $f_{\overline{z}}$ a.e.

Appendix 3. Let E and F be two complex Banach spaces (of finite or infinite dimension), and let U be a nonempty open subset (i.e. a domain) in E. A map $f : U \to F$ is *holomorphic* if and only if the complex derivative $d_x f(\lambda)$ (at $x \in U$ in the direction of $\lambda \in E$) defined as

$$d_x f(\lambda) = \lim_{t \to 0} \frac{f(x + t\lambda) - f(x)}{t} \in F$$

exists (in the norm of F) for each $(x, \lambda) \in U \times E$. This map is called the *Fréchet derivative* of f at x.

$f : U \to F$ is holomorphic if and only if for each $x \in U$, there is a continuous complex linear ($\mathbb{C}$-linear) map $D_x : E \to F$ such that

$$\frac{\|f(x + y) - f(x) - D_x(y)\|_F}{\|y\|_E} \to 0$$

as $y \to 0$ in E. In fact, $D_x = d_x f$.