HOLOMORPHIC FAMILIES OF RIEMANN SURFACES

Sudeb Mitra; CUNY/Queens College and Graduate Center

AMS Sectional Meeting #1215; Boston College,
Special Session on Automorphisms of Riemann Surfaces and Related Topics, III
March 29, 2026

Definition

A *simple holomorphic family* (W, π, B) consists of a pair of connected complex manifolds W and B, with B simply connected, and a surjective holomorphic map $\pi : W \to B$ satisfying the following conditions:

(i) First, the map π has "horizontally holomorphic" local trivializations. That means there is an open covering $\{B_\alpha\}$ of B, a topological space X, and homeomorphisms θ_α (the local trivializations) of $B_\alpha \times X$ onto $\pi^{-1}(B_\alpha)$ such that $\pi(\theta_\alpha(t, x)) = t$ for all $(t, x) \in B_\alpha \times X$ and for each x in X, the map $t \mapsto \theta_\alpha(t, x)$ from B_α into W is holomorphic.

(ii) Secondly, each fiber $\pi^{-1}(t)$ should be a Riemann surface.

A *morphism* of simple holomorphic families (W_1, π_1, B_1) to (W_2, π_2, B_2) is a pair of holomorphic maps $f : B_1 \to B_2$ and $g : W_1 \to W_2$ such that g restricts to each fiber as a bijective map of $\pi_1^{-1}(t)$ to $\pi_2^{-1}(f(t))$ for each t in B_1. If f and g are biholomorphic, the morphism is called an *isomorphism*. An isomorphism is an automorphism if $B_1 = B_2$ and $W_1 = W_2$.

Let $\Delta : \{z \in \mathbb{C} : |z| < 1\}$. Let $M(\Delta)$ denote the open unit ball of the complex Banach space $L^\infty(\Delta, \mathbb{C})$. Let G be a Fuchsian group acting freely on Δ. Let $M(G)$ be the open unit ball in the complex Banach space of Beltrami differentials for G. Recall that $\mu \in L^\infty(\Delta)$ with $\|\mu\|_\infty < 1$ is a Beltrami differential for G, if for every $g \in G$, we have

$$\mu(g(z))\frac{\overline{g'(z)}}{g'(z)} = \mu(z) \qquad \text{a. e. on } \Delta.$$

For each μ in $M(G)$ we denote by w^μ the unique quasiconformal map of the plane onto itself that satisfies the Beltrami equation $w_{\overline{z}} = \mu w_z$ in Δ, conformal in the exterior of Δ and satisfies

$$w^\mu(z) = z + \mathcal{O}(|z|^{-1}) \text{ as } z \to \infty.$$

We say that μ and ν in $M(G)$ are Teichmüller equivalent if $w^\mu(z) = w^\nu(z)$ when $|z| = 1$. The Teichmüller space $Teich(G)$ is the set of equivalence classes in $M(G)$. If $\mu \in M(G)$, we denote its equivalence class in $Teich(G)$ by $\Phi(\mu)$. It is well-known that $Teich(G)$ is a complex manifold so that the map $\Phi : M(G) \to Teich(G)$ is a holomorphic split submersion (which means that Φ is a base-point preserving, surjective, holomorphic map, with local holomorphic sections).

There exists a continuous map $\mathcal{S} : Teich(G) \to M(G)$ such that $\Phi \circ \mathcal{S} : Teich(G) \to Teich(G)$ is the identity map.

We call the map $\mathcal{S}$ the *Douady-Earle section* for the Teichmüller space $Teich(G)$.

Remark. The term *Douady-Earle section* was first used in the paper *Douady-Earle section, holomorphic motions, and some applications,* by Yunping Jiang and S.Mitra; Contemp. Math. **575**, (2012), 219–251.The real-analyticity of the Douady-Earle section is not explicitly stated in the paper *Conformally natural extensions of homeomorphisms of the circle*, by A. Douady and C. J. Earle; Acta Math. **157** (1986), 23-48. For universal Teichmüller space, the real-analyticity was proved in the paper *On holomorphic families of Riemann surfaces*, by C. J. Earle and A. Marden; Contemp. Math., **573**, (2012) 67–97; (the authors call it "barycentric section"). A self-contained proof of the real-analyticity of the Douady-Earle section for $Teich(G)$ is given in the paper *Teichmüller space of a closed set in the Riemann sphere*, by Xinlong Dong, Arshiya Farhath. G, and S. Mitra; to appear in Contemporary Mathematics (forthcoming).

Theorem 1. [Earle and Fowler, 1985] Let G be a Fuchsian group acting freely on Δ, and let $X = \Delta/G$. There is a simple holomorphic family (W, π, B) with the following properties:

(i) if (W', π', B') is a simple holomorphic family with some fiber $(\pi')^{-1}(t)$ biholomorphically equivalent to X, there is a morphism (f, g) of (W', π', B') to (W, π, B);

(ii) the above morphism (f, g) is unique up to an automorphism of (W, π, B).

These properties determine (W, π, B) up to isomorphism, and B is biholomorphically equivalent to $Teich(G)$.
Property (ii) means that if (f_1, g_1) and (f_2, g_2) are two morphisms of (W', π', B') to (W, π, B), there is an automorphism (ϕ, ψ) of (W, π, B) such that $f_2 = \phi \circ f_1$ and $g_2 = \psi \circ g_1$.

We define the Bers fiber space $F(G) \subset Teich(G) \times \mathbb{C}$ as follows:

$$F(G) = \{(\Phi(\mu), z) \in Teich(G) \times \mathbb{C}; \mu \in M(G) \text{ and } z \in w^{\mu}(\Delta)\}.$$

By a theorem of Bers, $F(G)$ is open in $Teich(G) \times \mathbb{C}$, and is therefore a complex manifold. If $g \in G$ and $\mu \in M(G)$, there is a unique Möbius transformation g^{μ} such that

$$g^{\mu} \circ w^{\mu} = w^{\mu} \circ g.$$

It is well-known that g^{μ} depends only on the Teichmüller class $\Phi(\mu)$, so we can write $g^{\mu} = g^{t}$ if $\Phi(\mu) = t$. We have

$$g^{t} \circ w^{\mu} = w^{\mu} \circ g \text{ if } g \in G \text{ and } \Phi(\mu) = t.$$

This implies that $g^{t}(w^{\mu}(\Delta)) = w^{\mu}(\Delta)$, and we can define an action of G on $F(G)$ as follows:

$$g(t, \zeta) = (t, g^{t}(\zeta)) \text{ for all } g \in G \text{ and } (t, \zeta) \in F(G).$$

By a theorem of Douady and Earle, the Teichmüller space $Teich(G)$ is simply connected. Also, G acts on the Bers Fiber space $F(G)$, producing a complex manifold $V(G) = F(G)/G$ and a holomorphic map $\pi : V(G) \to Teich(G)$. The complex manifold $V(G)$ is called the Teichmüller curve over $Teich(G)$. By a theorem of Earle and Fowler, the map π has horizontally holomorphic local trivializations. Thus, $(V(G), \pi, Teich(G))$ is a simple holomorphic family. Earle and Fowler proved that $(V(G), \pi, Teich(G))$ satisfies the conditions (i) and (ii) of Theorem 1.

The crucial point in the proof of Theorem 1 is the following important application of the λ-lemma.

Lemma. [Earle and Fowler] Let $\theta_\alpha : B_\alpha \times X \to \pi^{-1}(B_\alpha)$ be a local trivialization of (W, π, B) defined over the connected open set $B_\alpha \subset B$. If the map $\theta_\alpha(\cdot, x) : B_\alpha \to W$ is holomorphic for each x in X, then the map $\theta_\alpha(t, x) \mapsto \theta_\alpha(s, x)$ from the fibers $\pi^{-1}(t)$ to $\pi^{-1}(s)$ is quasiconformal for any fixed t and s in B_α.

Remark. Theorem 1 gives the *universal* property for all Teichmüller spaces. Its beauty is that it gives a characterization of Teichmüller spaces without mentioning quasiconformal mappings. Of course, lurking behind, are quasiconformal mappings, via the λ-lemma.

Here, it is worth quoting Earle and Fowler: “This theorem, which was already known for finite dimensional Teichmüller spaces (Grothendieck) provides the first characterization of the infinite dimensional Teichmüller spaces that makes no mention of quasiconformal or quasisymmetric mappings. Of course, this does not eliminate quasiconformal mappings from the theory. On the contrary, their role is as important as ever, but now, thanks to the λ-lemma, they enter as an inevitable consequence of other natural assumptions.”

Let X be a Riemann surface of finite type, with genus g and with n punctures $(2g-2+n>0)$. Then the Teichmüller space $T(g,n)$ is a complex manifold of dimension $3g-3+n$, whose points represent the Riemann surfaces of type (g,n). The Teichmüller curve $V(g,n)$ is a complex manifold with dimension $3g-2+n$ with a holomorphic projection $\pi_n : V(g,n) \to T(g,n)$ onto $T(g,n)$. The study of (global) holomorphic sections of the Teichmüller curves $\pi_n : V(g,n) \to T(g,n)$ was initiated by John H. Hubbard (1972) in his doctoral dissertation for the case $n=0$.

Theorem 2. [Hubbard (1972)] $\pi_0 : V(g,0) \to T(g,0)$ has no holomorphic sections if $g \geq 3$ and six if $g=2$.

Theorem 3. [Earle and Kra (1976)] The Teichmüller curve $\pi_n : V(g,n) \to T(g,n)$ has exactly n holomorphic sections if $g \geq 3$ and exactly $2n+6$ holomorphic sections if $g=2$.

The main theorem. (Mitra, 2026)
Let X be a hyperbolic Riemann surface, and let G denote its uniformizing Fuchsian group (so we have $X = \Delta/G$). For each x in X, the map $\pi : V(G) \to Teich(G)$ has a real-analytic section S_x. That means, the map $\pi : V(G) \to Teich(G)$ has a family of real-analytic sections $\{S_x : x \in X\}$, parametrized by the Riemann surface X.

$$\begin{tikzcd}
Teich(G') \arrow[r, "\alpha"] \arrow[dr, shift left, "\theta"] & F(G) \arrow[r, "\sigma"] \arrow[d, "f"] & V(G) \arrow[dl, shift right, "\pi"'] \\
& Teich(G) \arrow[ul, shift left, "\widehat{S}"] \arrow[ur, shift right, "S_x"'] &
\end{tikzcd}$$

Let $X = \Delta/G$, $X' = X \setminus \{x\} : x \in X$, and let G' be the torsion-free Fuchsian group uniformizing X'. We have

$$\alpha(\Phi(\mu)) = (\pi(\Phi(\mu)), \zeta) : \zeta \in w^{\mu}(\Delta).$$

Define $S_x = \sigma \circ \alpha \circ \widehat{S}$. The map α is biholomorphic, σ is holomorphic, and $\widehat{S}$ is real-analytic. Therefore, the map S_x is real-analytic. We have

$$\pi \circ S_x = \pi \circ \sigma \circ \alpha \circ \widehat{S} = f \circ \alpha \circ \widehat{S} = \pi \circ \widehat{S} = Id.$$

THANK YOU VERY MUCH!