Groups which act with almost all signatures

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X a compact Riemann surface of genus g.

Automorphism group G of X leads to covering map $X \to X/G$ branched at r places.

If X/G has genus h and those branch points have monodromy of order $m_1, \ldots, m_r$, respectively, then

$$[h; m_1, \ldots, m_r]$$

is the signature of the action of G on X.

Riemann's Existence Theorem

A finite group G acts on a compact Riemann surface X of genus g >1 if and only if there are elements of the group

$$a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r$$

which generate the group, satisfy the following equation,

$$\prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} c_j = 1_G$$

Generating vector: $(a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r)$

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and so that $m_j = |c_j|$ satisfy the Riemann Hurwitz formula

$$g = 1 + |G|(h-1) + \frac{|G|}{2} \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right).$$

Generating vector: $(a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r)$

For a fixed group G...

Potential signatures are those signatures $[h; m_1, \ldots, m_r]$ which satisfy the Riemann Hurwitz formula:

$$g = 1 + |G|(h-1) + \frac{|G|}{2}\sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right).$$

Actual signatures are those which also have a generating vector associated to them.

We write $\left[h; [n_1, t_1], \ldots, [n_s, t_s]\right]$ to mean the signature

$$[h; \underbrace{n_1, \ldots, n_1}_{t_1}, \ldots, \underbrace{n_s, \ldots, n_s}_{t_s}].$$

$\mathcal{O}(G) = \{\mathrm{Ord}(g) : g \in G\} - \{1\}$ is the order set.

With a very small number of exceptions, any signature of the form $\left[h; [n_1, t_1], \ldots, [n_s, t_s]\right]$ for $n_i \in \mathcal{O}(G)$ and $t_i \in \mathbb{Z}^+$ is a potential signature.

Easy to compute.

Potential signatures are those signatures $[h; m_1, \ldots, m_r]$ which satisfy the Riemann Hurwitz formula:

$$g = 1 + |G|(h-1) + \frac{|G|}{2}\sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right).$$

Actual signatures are those which also have a generating vector associated to them.

Hard to compute.

These are not always the same.

Example

The signature [0; 3,3,9] is a potential signature for $G = C_9$ since it satisfies Riemann-Hurwitz for a curve of genus 2 and a group of order 9 with elements of order 3 and 9.

But this signature cannot be an actual signature for *abelian* groups. There's an issue with the lcm of the m_i.

Sometimes they are badly not the same for a fixed group.

Example

Take $q = p^n$ for an odd prime p. Then $[0; \underbrace{2,\ 2, \ldots,\ 2}_{r>4}]$

is a potential signature for SL(2,q) but there is only one element of order 2 in this group.

That one element certainly doesn't generate the whole group!

Our Question

Which *groups* only have a finite number of potential signatures which fail to be actual signatures?

We say such groups act with almost all signatures (or are AAS).

Knowledge of automorphism groups and the corresponding monodromy has important applications:

- the study of the mapping class group
- inverse Galois theory
- Shimura varieties
- Jacobian varieties

$\mathcal{O}(G) = \{\mathrm{Ord}(g) : g \in G\} - \{1\}$ is the order set.

Theorem

A group G is AAS if and only if:

I. The commutator (or derived) subgroup $[G : G]$ contains an element of order every $n_i \in \mathcal{O}(G)$.

II. G may be generated by elements of order n_i for each $n_i \in \mathcal{O}(G)$.

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If **I.** is false then potential signatures $[h; n_i]$ for $h > 0$ are never actual signatures.

If **II.** is false then potential signatures $[0; \underbrace{n_i, n_i, \ldots, n_i}_{\geq 4}]$ are never actual signatures.

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II. G may be generated by elements of order n_i for each $n_i \in \mathcal{O}(G)$.

If **I.** is true, we exhibit generating vectors for any signature $[h; m_1, \ldots, m_r]$ with h beyond a certain bound.

If **II.** is true then we exhibit generating vectors for any signature $[h; m_1, \ldots, m_r]$ for r beyond a certain bound.

Example

Take a set $\{n_1, n_2, n_3\} \in \mathcal{O}(G)$ and $\{g_1, g_2, g_3\}$ set of generators of order n_3.

Say c_1 and $c_2 \in G$ with $o(c_i) = n_i$.

Example

Take a set $\{n_1, n_2, n_3\} \in \mathcal{O}(G)$ and $\{g_1, g_2, g_3\}$ set of generators of order n_3.

Say $c_1 = g_1 g_2^{-2} g_3$ and $c_2 = g_3^{-1} g_1 g_2 \in G$ with $o(c_i) = n_i$. Then

$$(c_1, c_2, \underbrace{g_2^{-1}, g_1^{-1}, g_3}_{c_2^{-1}}, \underbrace{g_3^{-1}, g_2, g_2, g_1^{-1}}_{c_1^{-1}})$$

is a generating vector for signature $[0; n_1, n_2, \underbrace{n_3, \ldots, n_3}_{7}]$.

Theorem

Any non-abelian finite simple group is AAS.

I. Non-abelian simple groups all have commutator subgroup the full group.

II. Take an element of order n_i. The set of conjugates of that element is a set of elements of order n_i and which generate a normal subgroup. Since simple, this is all of G.

Proposition

If a group G is AAS, then it is either a non-abelian p-group, or a perfect group.

A perfect group is one where the commutator subgroup is the whole group.

Since the commutator subgroup must contain elements of every order in $\mathcal{O}(G)$, any AAS group must be non-abelian.

Suppose not a p-group: p and q two distinct primes in $\mathcal{O}(G)$

Since AAS, G is generated by elements of order p which means $G/[G,G]$ is generated by elements of order p too.

But $G/[G,G]$ is abelian so $G/[G,G]$ is elementary abelian of order p^k for some k.

Same argument for the prime q implies $G/[G,G]$ is elementary abelian of order q^ℓ for some ℓ. So $G/[G,G]$ must be trivial, hence G is perfect.

Not all non-abelian p-groups are AAS.

Proposition

A non-abelian p-group of order p^n and exponent p^{n-1} is never AAS.

(the commutator subgroup is too small)

- Certainly all non-abelian p-groups of exponent p are AAS.
- We are working on classifying the p-group case.

- Certainly all non-abelian p-groups of exponent p are AAS.
- We are working on classifying the p-group case.

- There are perfect but non-simple groups which are AAS. For example the two perfect groups of order 960 and one of order 1080 are.
- From Magma: 229 perfect groups up to order 50,000, of which 26 are simple, and 86 are non-simple and AAS.

Suppose that $\left[h; [n_1, t_1], [n_2, t_2], \ldots, [n_i, t_i], \ldots, [n_r, t_r]\right]$ is an actual signature for a group G. Then the following are also actual signatures:

- $\left[h + 1; [n_1, t_1], [n_2, t_2], \ldots, [n_i, t_i], \ldots, [n_r, t_r]\right]$,
- $\left[h; [n_1, t_1], [n_2, t_2], \ldots, [n_i, t_i + 2], \ldots, [n_r, t_r]\right]$, and
- $\left[h; [n_1, t_1], [n_2, t_2], \ldots, [n_i, t_i + 1], \ldots, [n_r, t_r]\right]$ for n_i odd.

$$(a_1, b_1, \ldots a_h, b_h, c_{1,1}, \ldots c_{1,t_1}, \ldots, c_{i,1}, \ldots, c_{r,1}, \ldots, c_{r,t_r})$$

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- $\left[h; [n_1, t_1], [n_2, t_2], \ldots, [n_i, t_i+1], \ldots, [n_r, t_r]\right]$ for n_i odd.

$$(a_1, b_1, \ldots a_h, b_h, c_{1,1}, \ldots c_{1,t_1}, \ldots, c_{i,1}^2, c_{i,1}^{-1}, \ldots, c_{r,1}, \ldots, c_{r,t_r})$$

For each simple group up to order 10 000, what is the largest genus g so that there exists a potential signature for a curve of that genus which is, in fact, not an actual signature?

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Well, at least for covers of $\mathbb{P}^1$?

group	g	potential, but not actual signatures
PSL(2,7)	210	[0; 2, 2, 2, 3] [0; 2, 2, 2, 4] [0; 2, 2, 2, 2, 2]
$A_6 \cong$ PSL(2,9)	31	[0; 2, 2, 2, 3] [0; 3, 4, 4]
PSL(2,11)	56	[0; 2, 2, 2, 3]
PSL(2,16)	817	[0; 3, 3, 5] [0; 2, 5, 5] [0; 5, 5, 5] [0; 3, 5, 5]
PSL(2,25)	1821	[0; 2, 4, 5] [0; 2, 4, 6] [0; 2, 5, 5] [0; 2, 5, 6] [0; 2, 6, 6] [0; 3, 3, 5] [0; 3, 4, 4] [0; 3, 4, 6] [0; 3, 5, 5] [0; 3, 6, 6] [0; 4, 4, 5] [0; 4, 5, 6] [0; 5, 5, 5] [0; 5, 6, 6]

group	g	potential, but not actual signatures
A_7	3150	[0; 2, 3, 7] [0; 2, 4, 5] [0; 2, 4, 6] [0; 2, 5, 5] [0; 2, 5, 6] [0; 2, 6, 6] [0; 3, 3, 4] [0; 2, 2, 2, n_i] [0; 2, 2, 2, 2, 2]
M_{11}	9900?	[0; 2, 3, 8] [0; 2, 3,11] [0; 2, 4, 6] [0; 2, 4, 5] [0; 2, 4, 8] [0; 2, 5, 5] [0; 2, 5, 6] [0; 2, 6, 6] [0; 3, 3, 4] [0; 3, 3, 5] [0; 3, 3, 6] [0; 3, 3, 11] [0; 3, 4, 4] [0; 3, 5, 5] [0; 4, 4, 4] [0; 2, 2, 3, 3] [0; 2, 2, 2, 2, 2]
A_5, and PSL$(2,q)$ for $q = 8, 13, 17, 19, 23, 27$		Every potential signature is an actual signature. (Potential signature with $h = 0$.)

The End