Short homology bases for hyperelliptic hyperbolic surfaces

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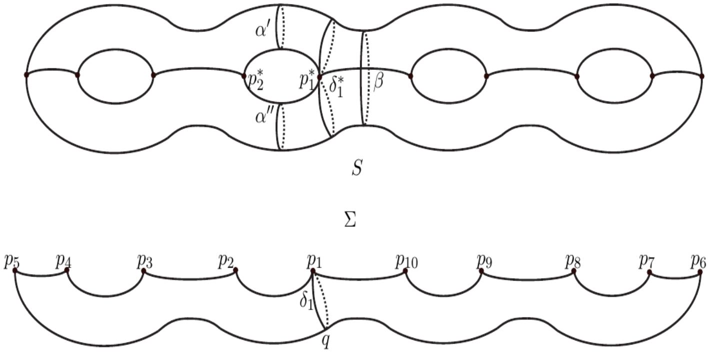


joint work with Peter Buser and Eran Makover

Result

Let S be a **hyperelliptic hyperbolic** surface S of genus $g \geq 2$.

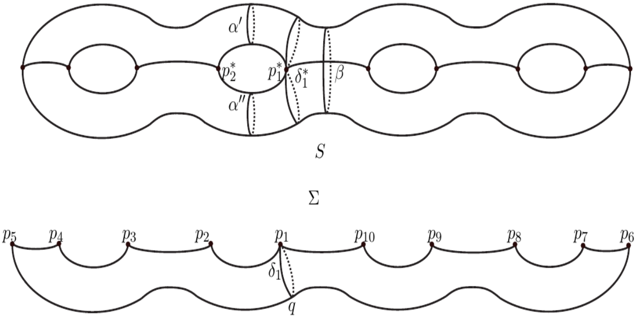
1.) Then S has almost $\frac{2}{3}g$ **homologically independent loops** of length **smaller** than a **constant** C_1.

2.) In the **lattice** of the **Jacobian torus** $J(S)$ of S there are almost $\frac{2}{3}g$ linearly independent **vectors** of length **smaller** than a **constant** C_2.

Overview

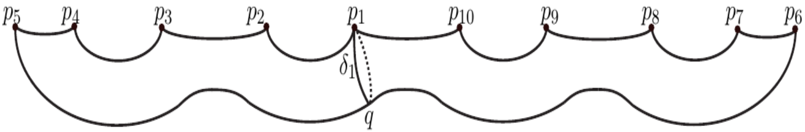
1. Short loops on hyperelliptic Riemann surfaces
2. Homology of edges and loops
3. A pruning algorithm
4. Jacobian of hyperelliptic surfaces

Short loops for hyperelliptic hyperbolic surfaces



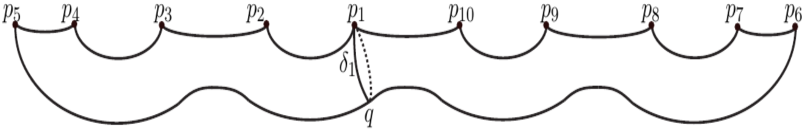
- A hyperelliptic surface S of genus $g \geq 2$ has a **hyperelliptic involution** $\phi : S \to S$.
- The **quotient** Σ is a **topological sphere** with $2g+2$ **cone points** of **angle** π.
- These **cone points** $\{p_i\}_{i=1,\ldots,2g+2}$ are the images of the **fixed points** under the natural projection $\Pi : S \to \Sigma$.
- We call the p_i^* as well as the p_i the **Weierstrass points** of S and Σ respectively.

Short loops for hyperelliptic hyperbolic surfaces



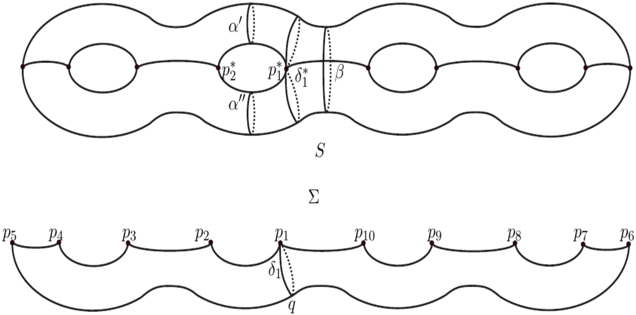
- We can find **short curves** on the **sphere** Σ by expanding disks around the **cone points**.
- The **upper bounds** for the lengths of these **curves** is due to an **area argument**.
- We obtain **short loops** on the surface S by lifting the short curves on Σ.

Short loops for hyperelliptic hyperbolic surfaces



- The curves we obtain on the sphere are either **edges** or **loops**.
- An **edge** lifts to a **simple closed geodesic** between **two** cone points on S.
- A **loop** lifts to a **figure-eight geodesic** connected to **one** cone point on S.
- In the latter case we **keep one of the loops** of the figure-eight geodesic.

Short loops for hyperelliptic hyperbolic surfaces

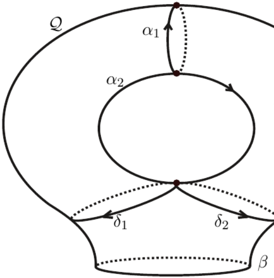


Lemma 1 Let S be a hyperelliptic hyperbolic surface, then there exist g non-separating loops $(\alpha_m)_{m=1,\dots,g}$ attached to Weierstrass points on S, such that

$$\ell(\alpha_m) \leq 4\log\left(2 \cdot \left(\frac{g-1}{g-m+2}\right) + 2\right), \text{ for all } m \in \{1,\dots,g\}.$$

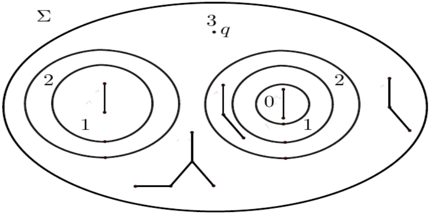
The median length ($m = \frac{g}{2}$) is bounded by $4\log(6)$ and the $(\alpha_m)_m$ are bounded by $4\log(g+1)$.

Problems

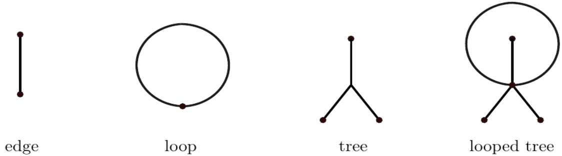


- Can two loops be in the same free homotopy class? **Yes.**
- Can several different loops together separate the surface? **Yes.**
- **Question:** How can we remove unwanted curves?

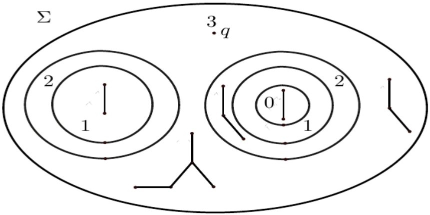
The graph $\mathcal{G}$ of edges and loops on the sphere Σ



- The **curves** we obtain on the sphere form a **graph** $\mathcal{G}$ on Σ.
- The graph $\mathcal{G}$ can have the following **connected components**:



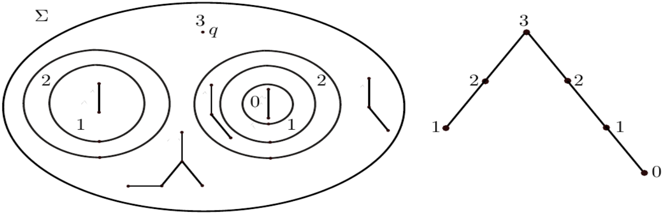
Homologically independent loops from a subgraph $\mathcal{H}$ of $\mathcal{G}$



- Let $\mathcal{H}$ be a subgraph of $\mathcal{G}$, obtained by **removing edges** or **loops**, but keeping the vertices / cone points.
- Let $\mathcal{H}^{\#}$ be the corresponding **lift**, with the modification for the figure-eight loops.

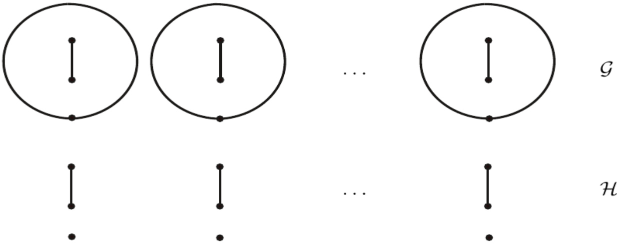
Lemma 2 $\mathcal{H}^{\#}$ in S is **non-separating** if and only if any **open connected component of** $\Sigma \setminus \mathcal{H}$ contains a **simple closed curve** Γ that separates $p_1, \ldots, p_{2g+2}$ into two odd subsets i.e. the **number of points on either side** of Γ is **odd**.

Pruning algorithm



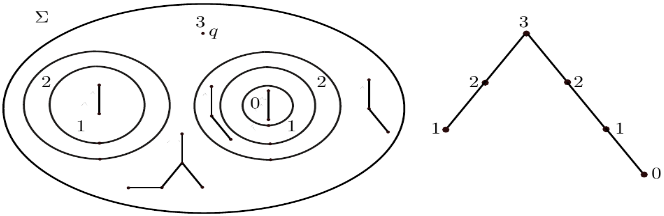
- **Idea 1: Edges** are "good", **loops** are "bad".
- **Idea 2: Divide** the sphere Σ along **loops** to get a **hierarchical subdivision** of regions of Σ.
- **Idea 3: Remove** curves from **bottom to top**.

Pruning algorithm



- The **worst case** is the case of a **loop with an edge inside** in Σ.
- For this case the ratio of **loops** to **cone points** in S is **2:3**.

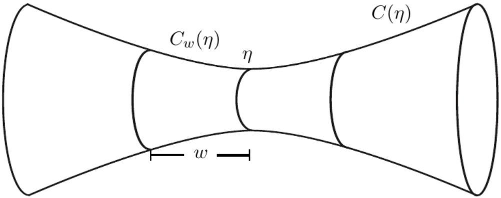
Pruning algorithm



Theorem 1 Let S be a hyperelliptic Riemann surface of genus $g \geq 2$. Then for any $\lambda \in (0,1)$ there exist $\lfloor \lambda \cdot \frac{2}{3} g \rfloor$ homologically independent loops $(\alpha_k)_{k=1,\ldots,\lfloor \lambda \cdot \frac{2}{3} g \rfloor}$, such that

$$\ell(\alpha_k) \leq C(\lambda) = 4 \log \left(\frac{6}{1-\lambda} + 2 \right) \text{ for all } k \in \{1, \ldots, \left\lfloor \lambda \cdot \frac{2}{3} g \right\rfloor \}.$$

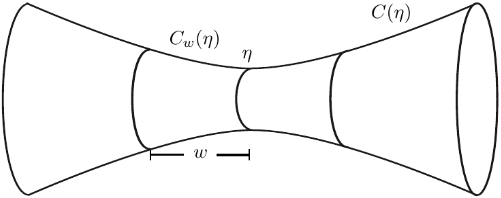
Lengths of vectors in the lattice of the Jacobian $J(S)$



By a result of Buser and Sarnak an **upper bound** on the **lengths** $\|v\|$ of a vector v in the **lattice** of the **Jacobian torus** $J(S)$ of a hyperbolic Riemann surface S can be obtained using **collars** around **non-separating loops**.

$$\|v\| \leq \frac{\ell(\eta)}{\pi - 2\arcsin\left(\cosh(w)^{-1}\right)}.$$

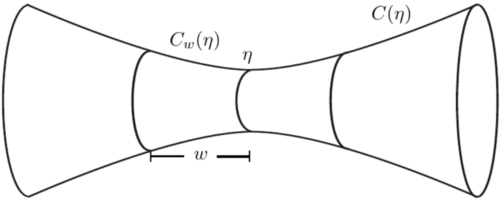
Collar lemma



The **Collar lemma** for hyperbolic surfaces states that every simple closed geodesic η in a hyperbolic surface S has a **collar** of at least width W where

$$W \geq \mathit{arcsinh}\left(\sinh\left(\frac{\ell(\eta)}{2}\right)^{-1}\right).$$

The Jacobian $J(S)$ of a hyperelliptic surface S



Combining the length estimates for our curves with the **Collar lemma**, we obtain

Theorem 2 Let S be a hyperelliptic hyperbolic surface of genus $g \geq 2$. Then for any $\lambda \in (0,1)$ there exist $\lfloor \lambda \cdot \frac{2}{3} g \rfloor$ linearly independent vectors $(v_k)_{k=1,\dots,\lfloor \lambda \frac{2}{3} g \rfloor}$ in the lattice of the Jacobian torus $J(S)$, such that for all $k \in \{1, \dots, \lfloor \lambda \frac{2}{3} g \rfloor\}$:

$$\|v_k\|^2 \leq \frac{C(\lambda)}{\pi - 2 \cdot \arcsin(\frac{1}{\cosh(w(\lambda)})} = D(\lambda)$$

Summary

Let S be a **hyperelliptic hyperbolic** surface S of genus $g \geq 2$.

1.) Then S has almost $\frac{2}{3}g$ **homologically independent loops** of length **smaller** than a **constant** C_1.

2.) In the **lattice** of the **Jacobian torus** $J(S)$ of S there are almost $\frac{2}{3}g$ linearly independent **vectors** of length **smaller** than a **constant** C_2.

Thank you for your attention!