Triangulations of Unorientable Surfaces

Preliminary Report

S. Allen Broughton[1] Eduardo Brandani da Silva[2]

[1] Rose-Hulman Institute of Technology (emeritus)

[2] Universidad Estatal de Maringá

20 Mar 2022

Sections

1. Overview, motivation and history, some introductory pictures
2. Triangular (quasi-platonic) group actions on surfaces
3. Symmetries - geometry and algebra
4. Search for symmetries and triangulated unorientable surfaces
5. Wrap up and questions

What are we looking for?

Suppose that

- M is closed, unorientable surface which can be triangulated by (l, m, n) triangles, and
- there is a “substantial” group of isometries of M that preserves the triangulation.

We wish to construct all (or at least many) such surfaces.

- We study S, the (orientation) double covering of M, the covering group $\langle \psi \rangle$, and the lifted triangulation.
- Now reverse the process. Find surfaces S, with a highly symmetric triangulation and a fixed point free anti-conformal involution ψ that preserves the triangulation. Then pick $M = S/\langle \psi \rangle$ and compute the automorphism group of the triangulation on M.

History

- The second author contacted the first author, to see if the well known of methods classifying triangulated (quasiplatonic) surfaces could be applied to unoriented surfaces.
- Desired application - next two slides
- The answer is yes and it appears that our comprehensive classification methodology is new. Comprehensive details and proofs will appear in [2] (under preparation).
- There is considerable literature on classification of symmetries for quasi-platonic surfaces. See some sample references on slide 43.
- Our methodology builds upon those articles.

Topological quantum codes - 1

- For more background on topological quantum computation see [1].
- Topological quantum computation uses a cellular decomposition or tiling of a compact 2-manifold.
- The quantum computer consists of qubits and check operators as in this table:

cellular component	purpose
edges	qubits
vertex	X_v vertex check operator
face	Z_f face check operator

- The vertex and face check operators only take into account the qubits, e, incident to a vertex or bounding a face.

Topological quantum codes -2

Topological quantum code models have grown increasingly more complex. See [1].

- the lattices of squares or hexagons on a torus.
- polyominos on a torus.
- cell decompositions of the projective plane.
- regular tessellations of compact orientable surfaces by regular polygons - these can be generated from triangulations of surfaces via dessins d'enfant.

Our current investigation seeks examples of unoriented surfaces to add to the list.

The (2,3,5)-tiled sphere - icosahedron

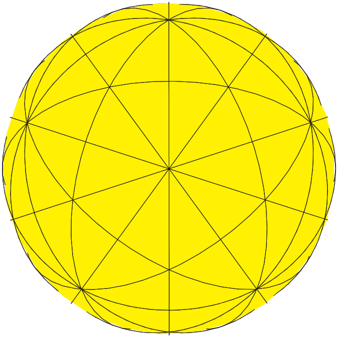

Figure 1: Sphere with (2,3,5) triangulation

The (2,4,4)-tiled torus

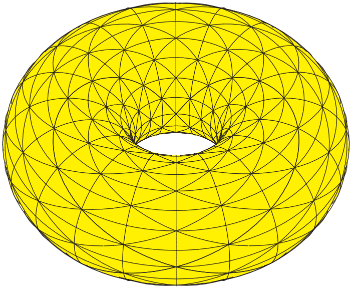

Figure 2: Torus with (2,4,4) triangulation

The unoriented quotient surfaces

- $(2,3,5)$ triangulation on the sphere (slide 7). The antipodal map
$$\psi(z) = \frac{-1}{\overline{z}}$$
preserves the triangulation and has no fixed points. We get a $(2,3,5)$ triangulation on the projective plane.
- $(2,4,4)$ triangulation on the torus (slide 8). Reflect a horizontal torus in a horizontal plane through the equator. Rotate torus a half turn around a vertical axis through the center of the donut hole. No point is fixed, though the triangulation is preserved. The quotient surface is the bottom half of the torus with two triangulated Möbius band sewn in with overlap, one for each oval of the bottom half of the torus.

Triangulations in the universal cover

- S is a closed Riemann surface of any genus.
- U is the universal cover of S, a simply connected geometry with constant curvature κ.
  - $U = \widehat{\mathbb{C}}$, Riemann sphere, $\kappa > 0$, genus of $S = 0$
  - $U = \mathbb{C}$, Euclidean plane, $\kappa = 0$, genus of $S = 1$
  - $U = \mathbb{H}$, hyperbolic plane, $\kappa < 0$, genus of $S \geq 2$
- (l, m, n) triangle: interior angles $\pi/l, \pi/m, \pi/n$ in counter-clockwise order, l, m, n integers ≥ 2 (slide 12)
- l, m, n satisfy

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 0 \Leftrightarrow \kappa > 0$$
$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 0 \Leftrightarrow \kappa = 0$$
$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 0 \Leftrightarrow \kappa < 0$$

Triangulation examples

- The (2,3,5)-tiled sphere (slide 7)
- The (4,4,3)-tiled hyperbolic plane (disk model)

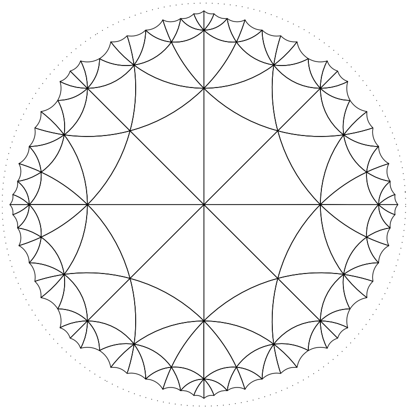

Figure 3: Hyperbolic plane with a (4,4,3) triangulation

The master tile

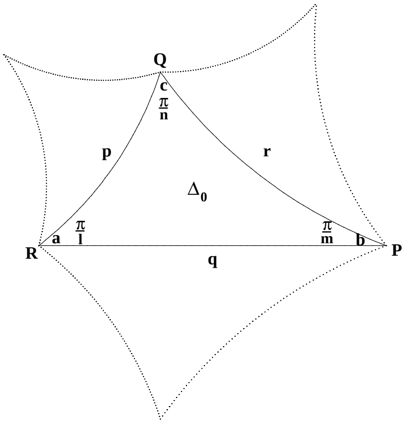

Figure 4: The master tile

More on the master tile

In the figure of the master tile Δ_0 and the example triangulations (slides 12,11):

- repeated reflections in the sides of tiles, starting with the master tile, create a triangulation $\mathcal{T}$ in U
- p, q, r denote the sides of Δ_0 and the reflections in those sides.
- Define
$$a = pq,\ \ b = qr,\ \ c = rp. \tag{1}$$
- a, b, c are counter-clockwise rotations centred at R, P, Q, respectively, and have orders l, m, n, respectively.

The triangle groups

- The full and orientation-preserving isometry groups in U preserving the triangulation $\mathcal{T}$ are the triangle groups
  - $T^*_{l,m,n} = \langle p, q, r \rangle$ (orientation-preserving or not), and
  - $T_{l,m,n} = \langle a, b, c \rangle$ (orientation preserving only).
- They have these presentations

$$T^*_{l,m,n} = \left\langle p, q, r : p^2 = q^2 = r^2 = (pq)^l = (qr)^m = (rp)^n = 1 \right\rangle,$$

$$T_{l,m,n} = \left\langle a, b, c : a^l = b^m = c^n = abc = 1 \right\rangle.$$

- Δ_0 is a fundamental domain for the $T^*_{l,m,n}$ action on U.
- the q-kite $\Delta_0 \cup q\Delta_0$ is a fundamental domain for the $T_{l,m,n}$ action on U, and the same for the p-kites and r-kites (see slides 12,11).

Uniformization of triangular actions

- Let $\pi_S : U \to S$ be the universal cover, with group of deck transformations Π.
- Π is a torsion free group of automorphism of U, isomorphic to $\pi_1(S)$.
- Assuming $\Pi < T_{l,m,n}$, then $S = U/\Pi$ inherits a triangulation $\overline{\mathcal{T}} = \mathcal{T}/\Pi$ by (l, m, n) triangles.
- Assuming $\Pi \triangleleft T_{l,m,n}$, then the finite group

$$G_S = T_{l,m,n}/\Pi$$

acts naturally upon $S = U/\Pi$ as a group of conformal automorphisms of S that preserves the triangulation $\overline{\mathcal{T}}$.

Group actions

- $\mathrm{Aut}(S)$ is the *group of conformal automorphisms* of S.
- A *conformal group action* of the finite group G on S is a monomorphism
$$\epsilon : G \to \mathrm{Aut}(S).$$
- Two actions $\epsilon_1, \epsilon_2 : G \to \mathrm{Aut}(S)$ are *algebraically equivalent* if and only if
$$\epsilon_2 = \epsilon_1 \circ \omega$$
for some $\omega \in \mathrm{Aut}(G)$.
- ϵ_1, ϵ_2 are algebraically equivalent if and only if they have the same image in $\mathrm{Aut}(S)$.

Surface kernel epimorphisms

- There is an exact sequence

$$\Pi \to T_{l,m,n} \to G_S \subseteq \mathrm{Aut}(S).$$

- If $\epsilon : G \to \mathrm{Aut}(S)$ is an action with $\epsilon(G) = G_S$, then the exact sequence $\Pi \to T_{l,m,n} \to G_S \xrightarrow{\epsilon^{-1}} G$ gives a so-called *surface kernel epimorphism*

$$\Pi \to T_{l,m,n} \xrightarrow{\eta} G, \tag{2}$$

giving us an alternative construction of a *triangular action* of G on the surface $S = \mathbb{H}/\Pi$.

- For $\omega \in \mathrm{Aut}(G)$, η, $\omega \circ \eta$ both define algebraically equivalent actions on the same surface $S = \mathbb{H}/\Pi$ (same kernel).

Generating triples - 1

- Given an action defined by (2), define

$$\overline{a} = \eta(a), \overline{b} = \eta(b), \overline{c} = \eta(c). \tag{3}$$

- The triple $(\overline{a}, \overline{b}, \overline{c})$ satisfies

$$G = \langle \overline{a}, \overline{b}, \overline{c} \rangle \tag{4}$$

$$\overline{a}^l = \overline{b}^m = \overline{c}^n = \overline{a}\overline{b}\overline{c} = 1. \tag{5}$$

- $(\overline{a}, \overline{b}, \overline{c})$ is called a *generating triple* with signature (l, m, n).

Generating triples - 2

- The elements $\overline{a}, \overline{b}, \overline{c}$ act by rotations at the corners of the master tile (slide 12).
- The signature is given by

$$(l, m, n) = (o(\overline{a}), o(\overline{b}), o(\overline{c})).$$

- The size of the group G and the genus σ of S satisfy the Riemann Hurwitz equation

$$\frac{2\sigma - 2}{|G|} = 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}. \tag{6}$$

Geometry of S/G

- Define
$$T = S/G = U/T_{l,m,n} \simeq \widehat{\mathbb{C}}.$$
- with quotient maps $\pi_G : S \to T$ and $\pi_T : U \to T$.
- π_G and π_T are branched over three points, say $\{0, 1, \infty\}$.
- The branch points are the images of vertices of the triangulations $\mathcal{T}$ or $\overline{\mathcal{T}}$.
- T has a triangulation or tiling $\overline{\overline{\mathcal{T}}}$, i.e., a system of vertices, edges, and faces, compatible with $\mathcal{T}$ and $\overline{\mathcal{T}}$. (slide 12)
    - vertices $\{0, 1, \infty\}$: images of three classes of vertices of tiles,
    - open edges $\{(-\infty, 0), (0, 1), (0, +\infty)\}$: homeomorphic images of three classes of open edges of tiles, and
    - open faces $\{H^+, H^-\}$: conformal images of two different classes of interiors of tiles.

What is a symmetry?

(See slide 8 to illustrate the concepts.)

- A symmetry ψ of a surface S is an anti-conformal involution, i.e., $\psi^2 = Id$.
- The *mirror* of ψ is the fixed point set $\mathcal{M}_\psi = \{x \in S : \psi(x) = x\}$.
- $\mathcal{M}_\psi$ is a possibly empty, disjoint set of closed, simple, geodesic curves called *ovals*.
- The quotient surface $S/\langle\psi\rangle$ is a Klein surface, orientable if $S\backslash\mathcal{M}_\psi$ is disconnected (*separating symmetry*), and is unorientable otherwise (*non-separating symmetry*).
- If ψ is fixed point free, then $S/\langle\psi\rangle$ is an (unorientable) Klein surface without boundary.
- **We shall search for fixed point free symmetries.**

Questions about symmetries

Assume S has a triangular action of G. We are interested in finding symmetries that normalize the action of G – and we shall assume so from now on. Questions that are typically asked about symmetries are (see refs 43):

1. Are there any symmetries ψ of S normalizing the G action?
2. If so, determine the G conjugacy classes of symmetries.
3. Is the mirror $\mathcal{M}_\psi$ non-empty, and if so, how many ovals are there?
4. Is ψ a separating symmetry?
5. Is the automorphism $\theta(g) = \theta_\psi(g) = \psi g \psi$ inner or outer?

Examples of symmetries

Example

Suppose S is defined by equations with real coefficients. Then complex conjugation in the ambient space defines a symmetry of S. The ovals are the components of the real curve. Specifically, the Fermat curve $x^n + y^n = \pm 1$ has an (n, n, n) triangulation. The symmetry is fixed point free for even n and choosing -1 for the right hand side.

Example

Suppose that the local reflection in the side of a tile extends to a globally defined isometry on S. Then the extended isometry is a symmetry. See master tile slide 12. The local reflection extends globally if there is a covering reflection on U that normalizes Π.

The quotient symmetry and symmetry type

Assume that the symmetry ψ normalizes the action of G on S.

- There is a quotient symmetry $\overline{\psi}$ on $T = S/G$.
- $\overline{\psi}$ permutes the branch points $\{0, 1, \infty\}$ and preserves $\overline{\overline{T}}$.
- $\overline{\psi}(z) = L(\overline{z})$, where $L(z)$ is linear fractional transformation determined by the permutation of $\{0, 1, \infty\}$ induced by $\overline{\psi}$.
- Here are the possibilities:

Type of ψ	permutation	Signature	$\overline{\psi}(z)$
I	$(0, 1, \infty)$	(l, m, n)	$\overline{z}$
II.a	$(1, 0, \infty)$	(l, l, n)	$1 - \overline{z}$
II.b	$(\infty, 1, 0)$	(l, m, l)	$\frac{1}{\overline{z}}$
II.c	$(0, \infty, 1)$	(l, m, m)	$\frac{\overline{z}}{\overline{z}-1}$

(7)

Standard symmetries

See slides 12,11 to illustrate the concepts following.

- The *standard Type I symmetries* of a triangular surface are the reflections ψ_p, ψ_q, ψ_r in the sides of the master tile, **if they exist.**
- Assume that $l = m$ so that the q-kite is also a rhombus. Let s be the rhombus bisector, i.e., the perpendicular bisector of q, which is the line segment from Q to qQ. Then the reflection ψ_s, is the *standard Type II.a symmetry* **if it exists.**
- Similar definitions apply to the p-kites and r-kites if the signatures permit.

More on symmetries, mirrors, and types

Proposition

A quotient symmetry $\overline{\psi}$ permutes the vertices, edges, and faces of $\overline{\overline{\mathcal{T}}}$. Hence, by lifting, ψ preserves $\overline{\mathcal{T}}$. In addition, the mirror of ψ is a union of tile edges (Type I) or rhombus bisectors (Type II).

Proposition

If a surface S has a symmetry of a given type (Table (7)) then it also has a symmetry of the same type with a non-empty mirror.

Proposition

Every symmetry with a non empty mirror is G conjugate to a standard symmetry of the same type.

Induced automorphism of a symmetry

- Given a symmetry, ψ, the map

$$\theta(g) = \theta_\psi(g) = \psi g \psi \tag{8}$$

is an automorphism satisfying

$$\theta^2 = Id \tag{9}$$

- For convenience, identify (a, b, c) with $(\overline{a}, \overline{b}, \overline{c})$, the generating triple of the G-action, via $a = \epsilon(\overline{a})$, etc.
- With this identification, θ is uniquely determined by the values $\theta(a), \theta(b), \theta(c)$ which are words in a, b, c.

Standard automorphism formulas

- p, q, r - edges of the master tile
- s, t, u - rhombus bisectors – if defined
- Automorphisms of the standard symmetries, if they exist, may be defined in terms of a, b, c by computing the local action on the master tile (slide 12).

Type	symmetry	$\theta(a), \theta(b), \theta(c)$
I	ψ_p	$c^{-1}a^{-1}c, b^{-1}, c^{-1}$
I	ψ_q	$a^{-1}, b^{-1}, bc^{-1}b^{-1}$
I	ψ_r	$a^{-1}, c^{-1}b^{-1}c, c^{-1}$
II.a	ψ_s	b^{-1}, a^{-1}, c^{-1}
II.b	ψ_t	c^{-1}, b^{-1}, a^{-1}
II.c	ψ_u	a^{-1}, c^{-1}, b^{-1}

(10)

Symmetry existence theorems - 1

Here is the first of two differently stated symmetry existence theorems. The first theorem was originally stated by Singerman [3] and assumes that the surface and action are given.

Theorem (Surface and action given)

Let (a, b, c) be a generating triple of signature (l, m, n) for a G action on S. Then S has a standard symmetry of Type I or II as given in Table (10) if and only if there is an automorphism θ of G satisfying the restrictions in the same Table. The automorphism θ is unique.

Symmetry existence theorems - 2

In our second theorem we assume a given G and θ and try to find a surface S and symmetry ψ.

Theorem (group and involution given)

Let θ be an automorphism of group G satisfying $\theta^2 = Id$. Then there is a surface S with standard symmetry ψ such that $\theta = \theta_\psi$ if and only if there is a generating vector (a, b, c) satisfying one of the formulas in Table (10). The signature and genus are given by the formulas on slide 19. There may be multiple solutions (a, b, c) and so the algebraic equivalence class of the action, the signature, and the genus of the surface may not be unique.

Companion symmetries

- Given two symmetries ψ, ψ', of the same type, there is a $g \in G$ such that

$$\psi' = \psi g = \psi \circ \epsilon(g)$$

  as isometries of S.
- We say that ψ, ψ' are companion symmetries. **They may not be conjugate!**
- Observe that

$$1 = (\psi')^2 = \psi g \psi g = \theta(g) g$$

  so

$$\theta(g) = g^{-1}.$$

- Since $p = qa^{-1}$ and $r = qb$, then ψ_p, ψ_q, and ψ_r are all companion symmetries.

Inverted elements, centralizer, and an action

- For $\theta \in \mathrm{Aut}(G)$ satisfying $\theta^2 = 1$, define

$$I_G(\theta) = \left\{ g \in G : \theta(g) = g^{-1} \right\}$$
$$Z_G(\theta) = \{ g \in G : \theta(g) = g \}$$

- Note that for $h \in G$

$$h\psi g h^{-1} = \psi\psi h\psi g h^{-1} = \psi\theta(h) g h^{-1}$$

- It is easily checked that $h \cdot g = \theta(h) g h^{-1}$ is a left G action on $I_G(\theta)$, equivalent to the conjugation action of G on symmetries of a fixed type. Call this action θ-twisted conjugation.
- This new action is defined even if ψ does not exist.

Finding Type I fixed point free symmetries

Theorem (Type I companion symmetries)

Suppose the triple (a, b, c) defines a triangular action of G upon S with Type I standard symmetry ψ_q. Let θ correspond to ψ_q. Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ be the orbits of the θ-twisted conjugation on $I_G(\theta)$.

Then, the conjugacy classes of Type I symmetries on S are in 1-1 correspondence with the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$. The orbits of the elements $\{1, a^{-1}, b\}$ determine the conjugacy classes of symmetries with fixed points. The remaining orbits, if any, correspond to equivalence classes of fixed point free Type I symmetries.

Finding Type II fixed point free symmetries

Theorem (Type II companion symmetries)

Suppose the triple (a, b, c) defines a triangular action of G upon S with isosceles signature (l, l, n) and Type II.a standard symmetry ψ_s. Let θ correspond to ψ_s. Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ be the orbits of the θ-twisted conjugation on $I_G(\theta)$.

Then the conjugacy classes of Type II.a symmetries on S are in 1-1 correspondence with the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$. The orbit of the element $\{1\}$ determines the conjugacy classes of symmetries with fixed points. The remaining orbits, if any, correspond to equivalence classes of fixed point free Type II.a symmetries.

Similar statements apply to Type II.b and Type II.c symmetries, signature permitting.

Automorphisms of M

Proposition (the group $\mathrm{Aut}(M)$)

Let G have a triangular action upon S, ψ a normalizing symmetry, and θ the corresponding automorphism. Then $Z_G(\theta)$ acts as a group of automorphisms of the Klein surface $M = S/\langle\psi\rangle$.

Remark

If θ is an inner automorphism then it is possible that ψ has a companion symmetry whose automorphism is the identity and in this case the entirety of G acts as automorphisms on M. This is not yet resolved.

Group and involutions

1. Select a finite group G.
2. Find $\mathrm{Aut}(G)$.
3. Find representatives of involution classes in $\mathrm{Aut}(G)$: $\theta_1, \ldots, \theta_d$.
4. For each θ in the list compute the following and then complete steps 5 and 6.

$$I_G(\theta) = \left\{ g \in G : \theta(g) = g^{-1} \right\},$$
$$Z_G(\theta) = \{ g \in G : \theta(g) = g \},$$
$$Z_{\alpha G}(\theta) = \{ \omega \in \mathrm{Aut}(G) : \omega\theta = \theta\omega \}.$$

5. Compute the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$ of the θ-twisted conjugation on $I_G(\theta)$.
6. Construct numbering map $on : I_G(\theta) \to \{1, \ldots, k\}$, where $on(g) = j \Leftrightarrow g \in \mathcal{O}_k$.

Type I

1. For $a, b \in I_G(\theta)$ check if $(a, b, (ab)^{-1})$ is a generating vector. If so, record in a list.
2. Find representatives of the $Z_{\alpha G}(\theta)$ - orbits in the list, this eliminates duplication.
3. Compute signature and genus for each resulting triple.
4. For each triple compute the following

$$((on(1), on(a^{-1}), on(b)), \{\text{remaining orbit numbers}\}).$$

The “remaining orbit numbers” correspond to fixed point free Type I symmetries.

Type II

First Type II.a

1. For $a \in G$ check if $(a, \theta(a^{-1}), (a\theta(a^{-1}))^{-1})$ is a generating vector. If so, record in a list.
2. Find representatives of the $Z_{\alpha G}(\theta)$ - orbits in the list, to eliminate redundancy.
3. Compute signature and genus for each resulting triple.
4. For each triple compute

$$(on(1), \{\text{remaining orbit numbers}\})$$

The remaining orbit numbers correspond to fixed point free Type II.a symmetries The process for Type II.b and Type II.c is similar.

Remarks

- Some of the analysis can be worked out by hand: cyclic, abelian, non-abelian groups of order *pq*, examples in the articles on the reference page (slide 43).
- We skip $\theta = Id$ since this only yields dihedral actions on the sphere.
- There may be some redundancy because of permuted signatures.
- Computations have been done for over 4,000 groups. Lets look at the record file for $Alt(5)$ (this is another file, not part of the slides).

Future work

- The methods here can be used for a detailed analysis of symmetries of n-gonal actions.
- The analysis of symmetries can be used to describe the moduli space of real curves as a subset of the moduli space of complex curves.

Q&A

Questions

Forthcoming works

- C.D. Albuquerque, E.B. da Silva, W.S Soares, *Quantum Codes for Topological Quantum Computation*, forthcoming.
- S.A. Broughton, E.B. da Silva *Triangulations of Unorientable Surfaces*, in preparation.

Some references on symmetries

- S.A. Broughton, E. Bujalance, A.F. Costa, J.M. Gamboa, G. Gromadzki, *Symmetries of Riemann surfaces on which PSL(2,q) acts as a Hurwitz automorphism group*, J. Pure and App. Alg., 106, 113-126.
- S.A. Broughton, E. Bujalance, A.F. Costa, J.M. Gamboa, G. Gromadzki, *Symmetries of Accola-Machlaclan and Kulkarni surfaces*, Proc. AMS, **127**(3)(1999), p. 637–646.
- D. Singerman, *Symmetries of Riemann surfaces with large automorphism group*, Math. Ann. **210** (1974), 17–32.
- E. Tyszkowska, *On Macbeath-Singerman symmetries of Belyi surfaces with PSL$(2, p)$ as a group of automorphisms*, Cent. Eur. J. Math. 1 (2003), no. 2, 208–220.